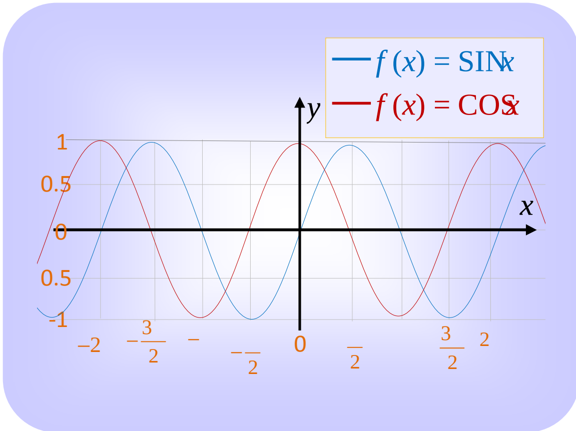
Unit 5



TRIGONOMETRIC FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *know principles and methods for sketching graphs of basic trigonometric functions.*
- *understand important facts about reciprocals of basic trigonometric functions.*
- *identify trigonometric identities.*
- *solve real life problems involving trigonometric functions.*

Main Contents

INTRODUCTION

IN MATHEMATICS, THE trigonometric functions (ALSO CALLED CIRCULAR FUNCTIONS) ARE FUNCTIONS OF ANGLES. THEY WERE ORIGINALLY USED TO RELATE THE ANGLES OF A T LENGTHS OF THE SIDES OF A TRIANGLE. LOOSELY TRANSLATED, trigonometry MEANS triangle measure. TRIGONOMETRIC FUNCTIONS ARE HIGHLY USEFUL IN THE STUDY OF TRIANGLE AND ALSO IN MANY DIFFERENT PHENOMENA IN REAL LIFE.

THE MOST FAMILIAR TRIGONOMETRIC FUNCTIONS ARE sine, cosine AND tangent. IN THIS UNIT, YOU WILL BE STUDYING THE PROPERTIES OF THESE FUNCTIONS IN DETAIL, INCLUDING AND SOME PRACTICAL APPLICATIONS. ALSO, YOU WILL EXTEND YOUR STUDY WITH AN TO THREE MORE TRIGONOMETRIC FUNCTIONS.

5.1 BASIC TRIGONOMETRIC FUNCTIONS

HISTORICAL NOTE:

Astronomy led to the development of trigonometry. The Greek astronomer **Hipparchus** (140 BC) is credited for being the originator of trigonometry. To aid his calculations regarding astronomy, he produced a table of numbers in which the lengths of chords of a circle were related to the length of the radius.

Hipparchus (190-120 BC)

Ptolomy, another great Greek astronomer of the time, extended this table in his major published work *Almagest* which was used by astronomers for the next 1000 years. In fact much of Hipparchus' work is known through the writings of Ptolomy. These writings found their way to Hindu and Arab scholars.

Aryabhata, a Hindu mathematician in the 6th century AD, drew up a table of the lengths of half-chords of a circle with radius one unit. Aryabhata actually drew up the first table of sine values.

In the late 16th century, Rhaeticus produced a comprehensive and remarkably accurate table of all the six trigonometric functions. These involved a tremendous number of tedious calculations, all without the aid of calculators or computers.

OPENING PROBLEM

FROM AN OBSERVER O, THE ANGLES OF ELEVATION OF THE BOTTOM AND THE TOP OF A FLAGPOLE ARE 36° AND 38° RESPECTIVELY. FIND THE HEIGHT OF THE FLAGPOLE.

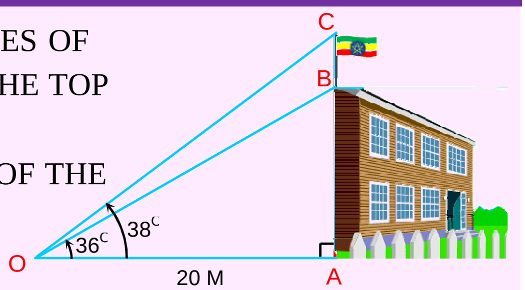


Figure 5.1

5.1.1 The Sine, Cosine and Tangent Functions

Basic terminologies

IF A GIVEN RAY $\overrightarrow{OA}$ (WRITTEN $\overrightarrow{OA}$) ROTATES AROUND A POINT O FROM ITS INITIAL POSITION TO A NEW POSITION, IT FORMS AN ANGLE AS SHOWN BELOW.

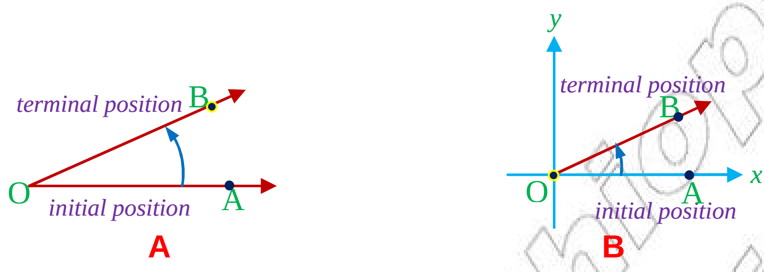


Figure 5.2

$\overrightarrow{OA}$ (INITIAL POSITION) IS CALLED THE **initial side** OF ∠.

$\overrightarrow{OB}$ (TERMINAL POSITION) IS CALLED THE **terminal side** OF ∠.

THE ANGLE FORMED BY A RAY ROTATING ANTICLOCKWISE IS TAKEN TO BE A POSITIVE ANGLE. AN ANGLE FORMED BY A RAY ROTATING CLOCKWISE IS TAKEN TO BE A NEGATIVE ANGLE.

EXAMPLE 1

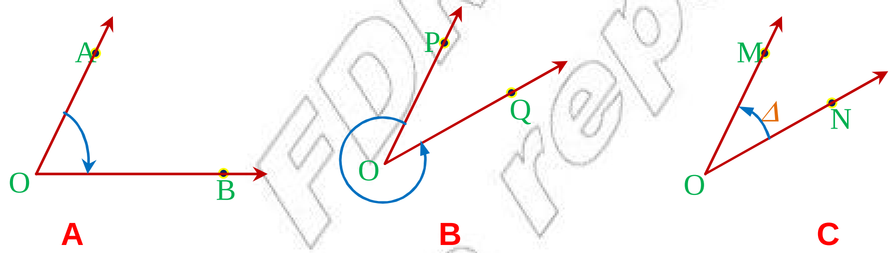


Figure 5.3

- ✓ ANGLE IN FIGURE 5.3A IS A NEGATIVE ANGLE WITH INITIAL SIDE $\overrightarrow{OA}$ AND TERMINAL SIDE $\overrightarrow{OB}$
- ✓ ANGLE IN FIGURE 5.3B IS A POSITIVE ANGLE WITH INITIAL SIDE $\overrightarrow{OP}$ AND TERMINAL SIDE $\overrightarrow{OQ}$
- ✓ ANGLE Δ IN FIGURE 5.3C IS A POSITIVE ANGLE WITH INITIAL SIDE $\overrightarrow{ON}$ AND TERMINAL SIDE $\overrightarrow{OM}$

Angles in standard position

AN ANGLE IN THE COORDINATE PLANE IS SAID TO BE IN **standard position**, IF

1 ITS VERTEX IS AT THE ORIGIN, AND

2 ITS INITIAL SIDE LIES ON THE POSITIVE x-AXIS.

EXAMPLE 2 THE FOLLOWING ANGLES ARE ALL IN STANDARD POSITION:

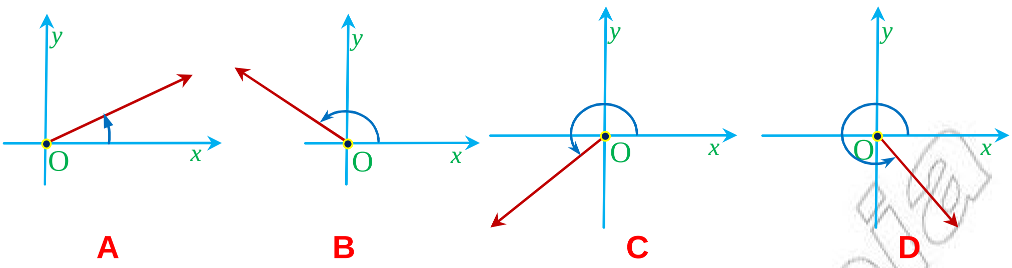


Figure 5.4

First, second, third and fourth quadrant angles

- IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION LIES IN THE FIRS THEN IT IS CALLED A first quadrant angle.
- IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION LIES IN THE SECO QUADRANT, THEN IT IS CALLED A second quadrant angle.
- IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION LIES IN THE THIR THEN IT IS CALLED A third quadrant angle.
- IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION LIES IN THE FOU QUADRANT, THEN IT IS CALLED A fourth quadrant angle.

EXAMPLE 3 THE FOLLOWING ARE ANGLES IN DIFFERENT QUADRANTS:

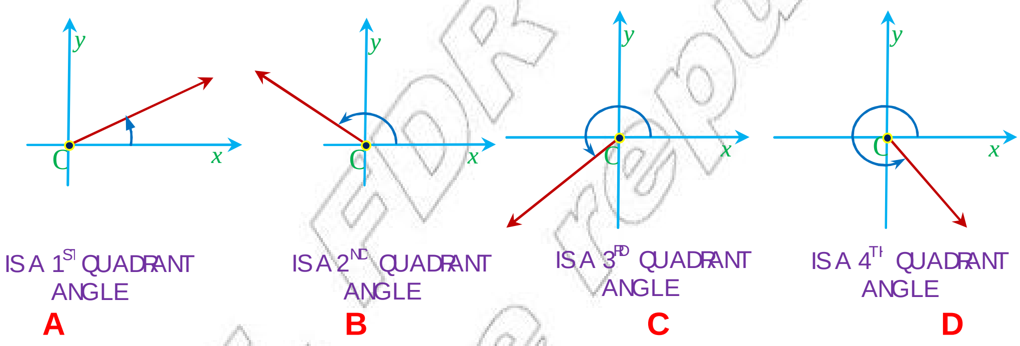
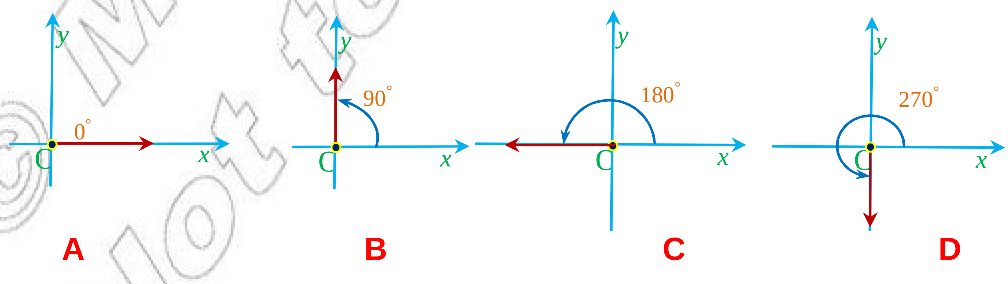


Figure 5.5

Quadrantal angles

IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION LIES ALONG THE x-AXIS OR THE y-AXIS, THEN THE ANGLE IS CALLED A quadrantal angle.

EXAMPLE 4 THE FOLLOWING ARE ALL QUADRANTAL ANGLES.



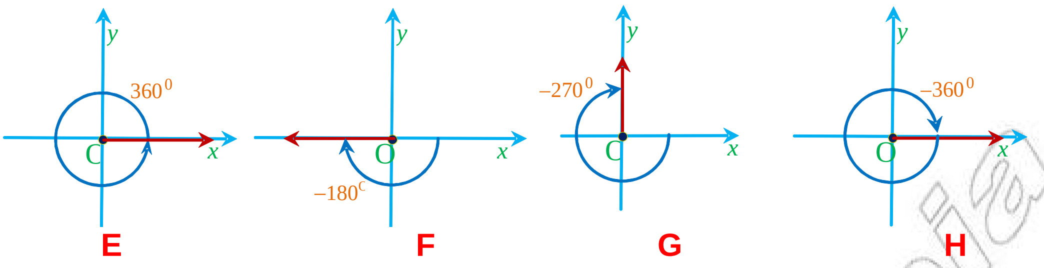


Figure 5.6

ANGLES WITH MEASURES OF –360°, –270°, –180°, –90°, 0°, 90°, 180°, 270°, 360° ARE EXAMPLES OF QUADRANTAL ANGLES BECAUSE THEIR TERMINAL SIDES LIE ALONG THE x-AXIS OR THE y-AXIS.

EXAMPLE 5 THE FOLLOWING ARE MEASURES OF DIFFERENT ANGLES. PUT THE ANGLES IN STANDARD POSITION AND INDICATE TO WHICH QUADRANT THEY BELONG:

A 200° **B** 1125° **C** – 900°

SOLUTION:

A $200^{\circ} = 180^{\circ} + 20^{\circ}$

∴ AN ANGLE WITH MEASURE 200° IS A THIRD QUADRANT ANGLE.

B $1125^{\circ} = 3(360)^{\circ} + 45^{\circ}$

1125° IS A MEASURE OF A FIRST QUADRANT ANGLE.

C $-900^{\circ} = 2(-360)^{\circ} + (-180^{\circ})$

– 900° IS A MEASURE OF A QUADRANTAL ANGLE.

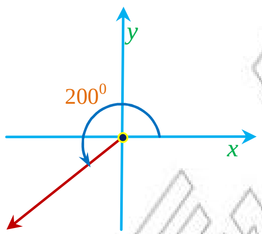


Figure 5.7

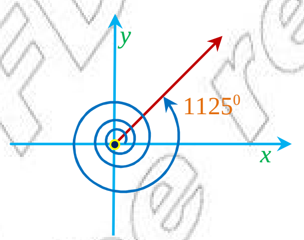


Figure 5.8

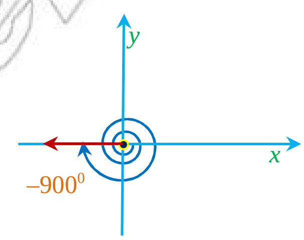


Figure 5.9

Exercise 5.1

THE FOLLOWING ARE MEASURES OF DIFFERENT ANGLES. PUT THE ANGLES IN STANDAR
INDICATE TO WHICH QUADRANT THEY BELONG:

A 240° **B** 350° **C** 620° **D** 666°

E –350° **F** – 480° **G** 550° **H** –1080°

Radian measure of angles

SO FAR WE HAVE MEASURED ANGLES IN DEGREES. HOWEVER, ANGLES CAN ALSO BE MEASURED IN RADIANS. SCIENTISTS, ENGINEERS, AND MATHEMATICIANS USUALLY WORK WITH ANGL

Group Work 5.1

1 DRAW A CIRCLE OF RADIUS 5 CM ON A SHEET OF PA

2 USING A THREAD MEASURE THE CIRCUMFERENCE C ND RECORD YOUR RESULT IN CENTIMETRES.

3 DIVIDE THE RESULT OBTAINED IN 2 BY 10 (LENGTH OF DIAMETER OF THE CIRCLE) AND GIVE YOUR ANSWER IN CENTIMETRES.

4 COMPARE THE ANSWER YOU OBTAINED IN 3 WITH THE VALUE OF π.

5 USING A THREAD, MEASURE AN ARC LENGTH OF 5 CM ON THE CIRCUMFERENCE OF THE CIRCLE AND NAME THE END POINTS A AND B AS SHOWN IN FIGURE 5.10.

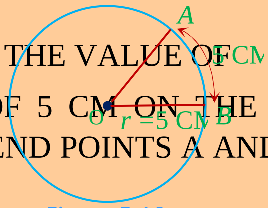


Figure 5.10

6 USING YOUR PROTRACTOR MEASURE ANGLE *AOB*.

7 IF YOU REPRESENT THE MEASURE OF THE CENTRAL ANGLE *AOB*, WHICH IS SUBTENDED BY AN ARC EQUAL IN LENGTH TO THE RADIUS AS 1 RADIAN, WHAT WILL BE THE APPROXIM 1 RADIAN IN DEGREES?

8 CAN YOU APPROXIMATE 180° AND 360° IN RADIANS?

9 DISCUSS YOUR FINDINGS AND FIND A FORMULA THAT CONVERTS DEGREE MEASUR MEASURE.

THE ANGLE SUBTENDED AT THE CENTRE OF A CIRCLE BY AN ARC EQUAL IN LENGTH TO T

1 *radian*. THAT IS $\theta = \frac{r}{r} = 1$ *radian*. (*See* FIGURE 5.11A)

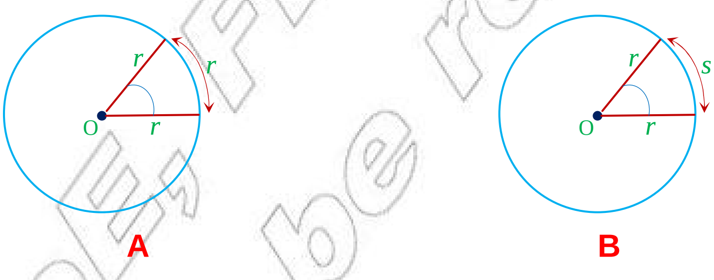


Figure 5.11

IN GENERAL, IF THE LENGTH OF THE ARC IS s UNITS AND THE RADIUS IS r UNITS, THEN $\theta = \frac{s}{r}$ RADIANS

(*See* FIGURE 5.11B) THIS INDICATES THAT THE SIZE OF THE ANGLE IS THE RATIO OF THE ARC TO THE LENGTH OF THE RADIUS.

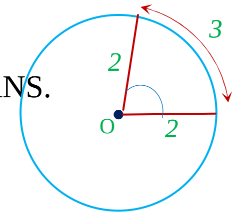


Figure 5.12

EXAMPLE 6 IF s = 3 CM AND r = 2 CM, CALCULATE θ IN RADIANS.

SOLUTION: $\theta = \frac{s}{r} = \frac{3}{2} = 1.5$ RADIA

EXAMPLE 7 CONVERT 360^{0} TO RADIANS.

SOLUTION: A CIRCLE WITH RADIUS R UNITS HAS CIRCUMFERENCE 2 r

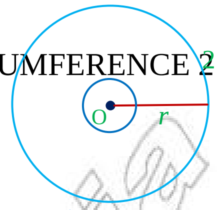


Figure 5.13

IN THIS CASE $\frac{s}{r}$ BECOMES $\frac{2\ \ r}{r} \Rightarrow\ = 2$

I.E., $360^{0} = 2$ RADIANS.

EXAMPLE 8 CAN YOU CONVERT 180^{0} TO *radian* MEASURE?

SOLUTION: SINCE $360° = 2$ RADIANS, $180° =$ RAD … *because* $180° = \frac{360^{0}}{2}$

IT FOLLOWS THAT 1 RAD $= \frac{180^{\circ}}{} = 57.3^{\circ}$

Rule 1

TO CONVERT DEGREES TO RADIANS, MULTIPLY BY $\frac{}{180^{o}}$

I.E., $radians = degrees \times \frac{}{180^{o}}$.

EXAMPLE 9

A CONVERT 30^{O} TO RADIANS. **B** CONVERT 240^{O} TO RADIANS.

SOLUTION:

A $30^{O} = 30° \times \frac{}{180°} = \frac{}{6}$ RADIAN.

B $240^{O} = 240 \times \frac{}{180} = \frac{4}{3}$ RADIANS.

Rule 2

TO CONVERT RADIANS TO DEGREES, MULTIPLY BY $\frac{180^{o}}{}$

I.E., $degrees = radians \times \frac{180^{o}}{}$.

EXAMPLE 10

A $\frac{}{2}$ RAD $= \frac{}{2} \times \frac{180^{O}}{} = 90^{0}$

B -4 RAD $= -4 \times \frac{180^{o}}{} = -720^{0}$

Exercise 5.2

1 CONVERT EACH OF THE FOLLOWING TO *radians*:

A 60 **B** 45 **C** –150 **D** 90 **E** –270 **F** 135

2 CONVERT EACH OF THE FOLLOWING *radians* TO *degrees*:

A $\frac{}{12}$ **B** $-\frac{}{6}$ **C** $\frac{2}{3}$ **D** $\frac{5}{6}$ **E** $-\frac{10}{3}$ **F** 3

Definition of the sine, cosine and tangent functions

THE Sine, Cosine and Tangent Functions ARE THE THREE basic trigonometric functions.

TRIGONOMETRIC FUNCTIONS WERE ORIGINALLY USED TO RELATE THE ANGLES OF A TRIANGL LENGTHS OF THE SIDES OF A TRIANGLE. IT IS FROM THIS PRACTICE OF MEASURING T TRIANGLE WITH THE HELP OF ITS ANGLES (OR VICE VERSA) THAT THE NAME TRIGONOME

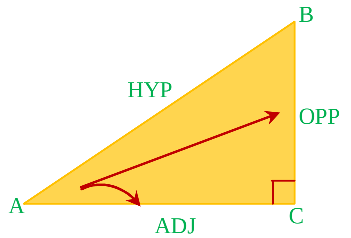


Figure 5.14

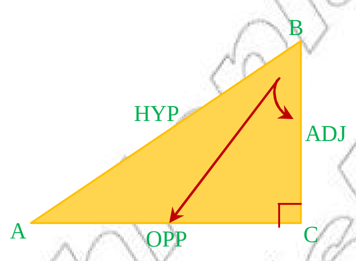


Figure 5.15

LET US CONSIDER THE RIGHT ANGLED TRIANGLES IN FIGURE 5.14 AND FIGURE 5.15

YOU ALREADY KNOW THAT, FOR A GIVEN RIGHT ANGLED TRIANGLE, THE hypotenuse (HYP SIDE WHICH IS OPPOSITE THE RIGHT ANGLE AND IS THE LONGEST SIDE OF THE TRIANGLE

FOR THE ANGLE MARKED BY (See FIGURE 5.14)

- ✓ $\overline{BC}$ IS THE SIDE opposite (OPP) ANGLE .
- ✓ $\overline{AC}$ IS THE SIDE adjacent (ADJ) ANGLE .

SIMILARLY, FOR THE ANGLE MARKED BY (See FIGURE 5.15)

- ✓ $\overline{AC}$ IS THE SIDE opposite (OPP) ANGLE .
- ✓ $\overline{BC}$ IS THE SIDE adjacent (ADJ) ANGLE .

Definition 5.1

If is an angle in standard position and $P(a,b)$ is a point on the terminal side of , other than the origin $\mathbf{O}(0, 0)$, and r is the distance of point P from the origin $\mathbf{O}$, then

$$\sin = \frac{OPP}{HYP} = \frac{b}{r}$$

$$\cos = \frac{ADJ}{HYP} = \frac{a}{r}$$

$$\tan = \frac{OPP}{ADJ} = \frac{b}{a}$$

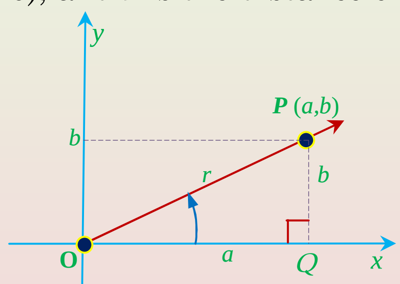


Figure 5.16

REMEMBER THAT OPQ IS A RIGHT ANGLE TRIANGLE.

(BY THE PYTHAGORAS THEOREM, $r = \sqrt{a^2 + b^2}$)

(SIN , COS AND TAN ARE ABBREVIATIONS OF SINE, COSINE AND TANGENT, RESPECTIVELY.)

TRIGONOMETRIC FUNCTIONS CAN BE CONSIDERED IN THE SAME WAY AS ANY GENERAL LINEAR, QUADRATIC, EXPONENTIAL OR LOGARITHMIC.

THE INPUT VALUE FOR A TRIGONOMETRIC FUNCTION IS AN angle. THAT ANGLE COULD BE MEASURED IN DEGREES OR RADIANS. THE OUTPUT VALUE FOR A TRIGONOMETRIC FUNCTION IS A pure number WITH NO UNIT.

EXAMPLE 11 IF IS AN ANGLE IN STANDARD POSITION AND P (3, 4) IS A POINT ON THE TERMINAL SIDE OF , THEN EVALUATE THE SINE, COSINE AND TANGENT OF .

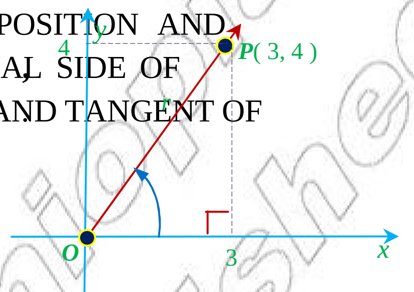


Figure 5.17

SOLUTION: THE DISTANCE $r = \sqrt{3^2 + 4^2} = 5$ UNITS

SO $\text{SIN} = \frac{OPP}{HYP} = \frac{4}{5}$ $\text{COS} = \frac{ADJ}{HYP} = \frac{3}{5}$ AND

$\text{TAN} = \frac{OPP}{ADJ} = \frac{4}{3}$.

Exercise 5.3

EVALUATE THE SINE, COSINE AND TANGENT FUNCTIONS OF IF IS IN STANDARD POSITION AND ITS TERMINAL SIDE CONTAINS THE GIVEN POINT P (x, y):

A P (3, – 4) **B** P (– 6, – 8) **C** P (1, –1)

D P $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **E** P $\left(4\sqrt{5}, -2\sqrt{5}\right)$ **F** P (1, 0)

The unit circle

THE CIRCLE WITH CENTRE AT (0, 0) AND RADIUS 1 UNIT IS CALLED THE unit circle.

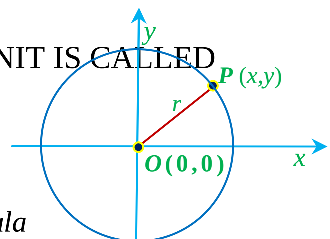


Figure 5.18

CONSIDER A POINT P (x, y) ON THE CIRCLE (See FIGURE 5.18)

SINCE OP = R, THEN $\sqrt{(x-0)^2 + (y-0)^2} = r$ … *by distance formula*

$\therefore x^2 + y^2 = r^2$ … *squaring both sides*

WE SAY THAT $x^2 + y^2 = r^2$ IS THE EQUATION OF A CIRCLE WITH CENTRE (0, 0) AND RADIUS R. ACCORDINGLY, THE EQUATION OF THE *unit circle* IS $x^2 + y^2 = 1$. (AS $r = 1$)

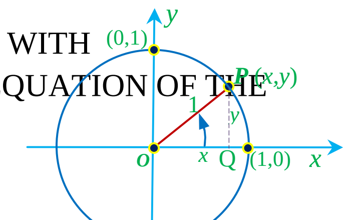


Figure 5.19

LET THE TERMINAL SIDE OF INTERSECT THE *unit circle* AT POINT P (x, y). SINCE $r = x^2 + y^2 = 1$, THE sine, cosine AND tangent FUNCTIONS OF ARE GIVEN AS FOLLOWS:

$$\text{SIN } \theta = \frac{OPP}{HYP} = \frac{y}{r} = \frac{y}{1} = y \quad \text{... the } y\text{-coordinate of } P$$

$$\text{COS } \theta = \frac{ADJ}{HYP} = \frac{x}{r} = \frac{x}{1} = x \quad \text{... the } x\text{-coordinate of } P$$

$$\text{TAN } \theta = \frac{OPP}{ADJ} = \frac{y}{x}$$

EXAMPLE 12 USING THE UNIT CIRCLE, FIND THE VALUES OF THE sine, cosine AND tangent OF θ; IF $\theta = 90^{O}, 180^{O}, 270^{O}$.

SOLUTION: AS SHOWN IN FIGURE 5.20, THE TERMINAL SIDE OF THE ANGLE 90^{O} INTERSECTS THE UNIT CIRCLE AT (0, 1). SO (x, y) = (0, 1).

HENCE, SIN 90^{O} = y = 1, COS 90^{O} = x = 0 AND TAN 90^{O} IS UNDEFINED SINCE $\frac{y}{x} = \frac{1}{0}$

THE TERMINAL SIDE OF THE ANGLE 180^{O} INTERSECTS THE UNIT CIRCLE AT (–1, 0). (*See* FIGURE 5.21) SO, (x, y) = (–1, 0).

HENCE, SIN 180^{O} = y = 0, COS 180^{O} = x = –1 AND TAN 180^{O} = $\frac{y}{x} = \frac{0}{-1} = 0$.

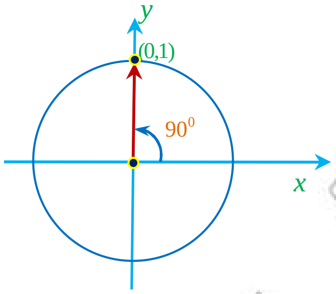


Figure 5.20

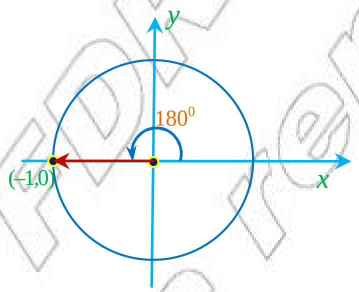


Figure 5.21

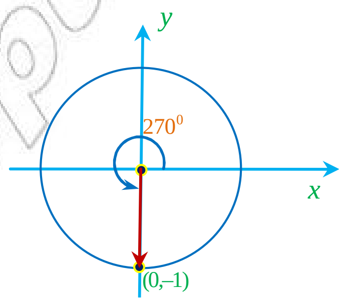


Figure 5.22

THE TERMINAL SIDE OF THE ANGLE 270^{O} INTERSECTS THE UNIT CIRCLE AT (0, –1). (*See* FIGURE 5.22) SO (x, y) = (0, –1). HENCE, SIN 270^{O} = y = –1, COS 270^{O} = x = 0 AND TAN 270^{O} IS UNDEFINED SINCE $\frac{y}{x} = \frac{-1}{0}$.

Exercise 5.4

1 USING THE UNIT CIRCLE, FIND THE VALUES OF THE SINE, COSINE AND TANGENT FU THE FOLLOWING QUADRANTAL ANGLES:

A 0^{O} **B** 360^{O} **C** 450^{O}

D 540^{O} **E** 630^{O}

Trigonometric values of 30°, 45° and 60°

THE FOLLOWING GROUP WORK WILL HELP YOU TO FIND THE TRIGONOMETRIC VALUES OF THE S ANGLE 45°.

Group Work 5.2

CONSIDER THE ISOSCELES RIGHT ANGLE TRIANGLE IN FIG

A CALCULATE THE LENGTH OF THE HYPOTENUSE A

B FROM THE PROPERTIES OF AN ISOSCELES RIGHT ANGLE TRIANGLE WHAT IS THE MEASURE OF ANGLE A?

C ARE THE ANGLES A AND B CONGRUENT?

D WHICH SIDE IS OPPOSITE TO ANGLE A?
WHICH SIDE IS ADJACENT TO ANGLE A?

E FIND SIN A, COS A AND TAN A.

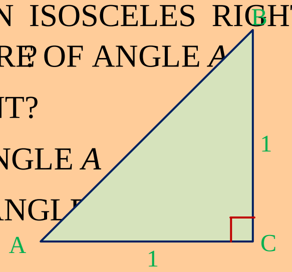


Figure 5.23

FROM GROUP WORK 5.2 YOU HAVE FOUND THE VALUES OF SIN 45°, COS 45° AND TAN 45°.

ANOTHER WAY OF FINDING THE TRIGONOMETRIC VALUES OF 45° IS TO PLACE THE ANGLE 45° IN STANDARD POSITION AS SHOWN IN FIGURE 5.24.

WHEN WE PLACE THE ANGLE 45° IN STANDARD POSITION, ITS TERMINAL SIDE INTERSECTS THE U CIRCLE AT $P(x, y)$.

TO CALCULATE THE COORDINATES OF P, DRAW $\overline{PD}$ PARALLEL TO THE y-AXIS.

ΔOPD IS AN ISOSCELES RIGHT ANGLE TRIANGLE.

BY PYTHAGORAS RULE, $(OD)^2 + (PD)^2 = (OP)^2$

SINCE $OD = PD$, $(PD)^2 + (PD)^2 = (OP)^2$.

THAT IS, $y^2 + y^2 = 1^2 \Rightarrow 2y^2 = 1 \Rightarrow y^2 = \frac{1}{2}$

$$\Rightarrow y = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} .$$

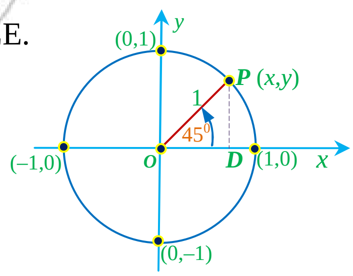


Figure 5.24

SINCE THE TRIANGLE IS ISOSCELES, BOTH THE x- AND y-COORDINATES ARE THE SAME.

THEREFORE THE TERMINAL SIDE OF THE ANGLE 45° INTERSECTS THE UNIT CIRCLE AT $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

HENCE, SIN 45° = $y = \frac{\sqrt{2}}{2}$; COS 45° = $x = \frac{\sqrt{2}}{2}$ AND TAN 45° = $\frac{y}{x} = \frac{\left(\frac{\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)} = 1$

Trigonometric values for 30° and 60°

CONSIDER THE EQUILATERAL TRIANGLE ABC IN FIGURE 5.25, WITH SIDE LENGTH 2 UNITS. THE ALTITUDE $\overline{BD}$ BISECTS $\angle B$ AS WELL AS SIDE $\overline{AC}$. HENCE $\angle ABD = 30^{O}$ AND $AD = 1$ (HALF OF THE LENGTH OF AC).

BY PYTHAGORAS THEOREM, THE LENGTH OF THE ALTITUDE IS h WHERE

$$h^2 + 1^2 = 2^2 \quad \Rightarrow h^2 = 4 - 1 = 3 \quad \Rightarrow h = \sqrt{3}$$

NOW IN THE RIGHT-ANGLED TRIANGLE ABD,

$$\text{SIN } 30^{O} = \frac{1}{2} = 0.5 \qquad \text{SIN } 60^{O} = \frac{\sqrt{3}}{2}$$

$$\text{COS } 30^{O} = \frac{\sqrt{3}}{2} \qquad \text{COS } 60^{O} = \frac{1}{2} = 0.5$$

$$\text{TAN } 30^{O} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \text{TAN } 60^{O} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

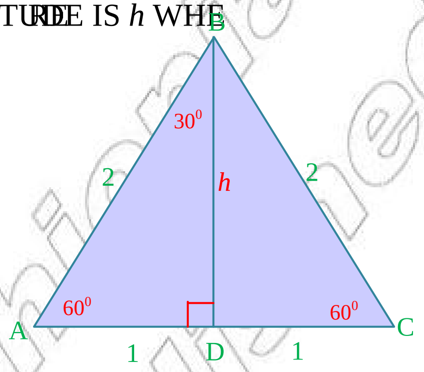


Figure 5.25

Trigonometric values of negative angles

Remember that AN ANGLE IS positive IF MEASURED ANTICLOCKWISE AND negative IF CLOCKWISE.

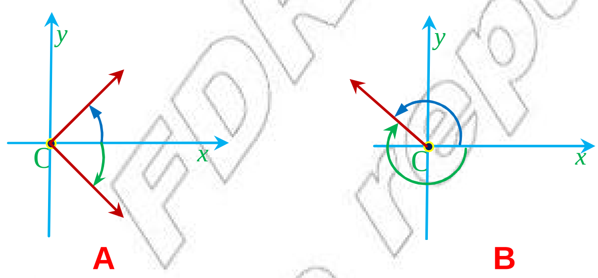


Figure 5.26

IS A POSITIVE ANGLE WHEREAS IS A NEGATIVE ANGLE.

EXAMPLE 13 USING THE UNIT CIRCLE, FIND THE VALUES OF THE SINE, COSINE AND TANGENT FUNCTIONS OF WHEN $= -180^{O}$.

THE TERMINAL SIDE OF -180^{O} INTERSECTS THE UNIT CIRCLE AT $(-1, 0)$. SO $(x, y) = (-1, 0)$.

HENCE, SIN $(-180^{O}) = y = 0$,

COS $(-180^{O}) = x = -1$

AND TAN $(-180^{O}) = \dfrac{y}{x} = \dfrac{0}{-1} = 0$.

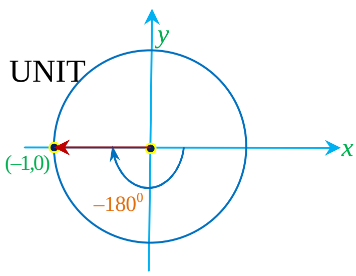


Figure 5.27

EXAMPLE 14 USING THE UNIT CIRCLE, FIND THE VALUES OF THE SINE, COSINE AND TANG FUNCTIONS OF WHEN $= -45^{O}$.

SOLUTION: PLACE THE -45° ANGLE IN STANDARD POSITION. ITS TERMINAL SIDE INTERSECTS THE UNIT CIRCLE AT Q (x, y).

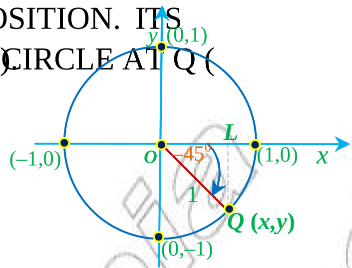


Figure 5.28

TO DETERMINE THE COORDINATES OF Q, DRAW $\overline{QL}$ PARALLEL TO THE y-AXIS.

ΔOQL IS AN ISOSCELES RIGHT TRIANGLE.

BY PYTHAGORAS THEOREM, $(OL)^2 + (QL)^2 = (OQ)^2$

SINCE $OL = QL$, $(QL)^2 + (QL)^2 = (OQ)^2$.

THAT IS $y^2 + y^2 = 1^2 \Rightarrow 2y^2 = 1 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm\sqrt{\frac{1}{2}}$

$\therefore y = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$. … ***Remember that*** *y is negative in the fourth quadrant*

SINCE THE TRIANGLE IS ISOSCELES, $QL = OL = \frac{\sqrt{2}}{2}$

THEREFORE, THE x-COORDINATE OF Q IS $\frac{\sqrt{2}}{2}$ … *Note that x is positive in the fourth quadrant*

SO, THE TERMINAL SIDE OF THE -45° ANGLE INTERSECTS THE UNIT CIRCLE AT P$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

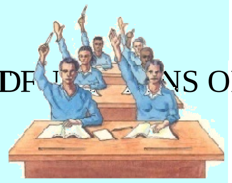
I.E., $(x, y) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

HENCE, SIN $(-45^{\circ}) = y = \frac{\sqrt{2}}{2}$; COS $(-45^{\circ}) = x = \frac{\sqrt{2}}{2}$ AND TAN $(-45^{\circ}) = \frac{y}{x} = \frac{\left(-\frac{\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)} = -1$.

OBSERVE THAT FROM THE TRIGONOMETRIC VALUES OF 45° AND -45°,

SIN (-45°) = –SIN 45°, COS (-45°) = COS 45° AND TAN(-45°) = –TAN45°.

ACTIVITY 5.1



1 FIND THE VALUES OF THE SINE, COSINE AND TANGENT FUNCTIONS OF … AND COMPLETE THE FOLLOWING TWO TABLES:
(USE A DASH "–" IF IT IS UNDEFINED).

	0^{o}	30^{o}	45^{o}	60^{o}	90^{o}	180^{o}	270^{o}	360^{o}
sin	0				1		–1	
cos						–1		
tan					–			

	–30°	–45°	–60°	–90°	–180°	–270°	–360°
SIN	$-\frac{1}{2}$		$-\frac{\sqrt{3}}{2}$				
COS	$\frac{\sqrt{3}}{2}$		$\frac{1}{2}$	0			
TAN	$-\frac{\sqrt{3}}{3}$		$-\sqrt{3}$	–			

2 WHICH OF THE FOLLOWING PAIRS OF VALUES ARE EQUAL?

- **A** SIN(–30°) AND SIN(30°)
- **B** COS(–30°) AND COS(30°)
- **C** TAN(–30°) AND TAN(30°)
- **D** SIN(– 45°) AND SIN(45°)
- **E** COS(– 45°) AND COS(45°)
- **F** TAN(– 45°) AND TAN(45°)
- **G** SIN(– 60°) AND SIN(60°)
- **H** COS(– 60°) AND COS(60°)
- **I** TAN(– 60°) AND TAN(60°)

3 HOW DO YOU COMPARE THE VALUES OF:

- **A** SIN (–) AND SIN
- **B** COS (–) AND COS
- **C** TAN (–) AND – TAN

FROM ACTIVITY 5.1 YOU CONCLUDE THE FOLLOWING:

IF IS ANY ANGLE, THEN SIN(–) = –SIN, COS(–) = COS and TAN(–) = –TAN.

LET US REFER TO FIGURE 5.29 TO JUSTIFY THE ABOVE.

$$\text{SIN} = \frac{y}{r},\ \text{SIN}(-\ ) = \frac{-y}{r} = -\left(\frac{y}{r}\right) \therefore \text{SIN}(-\ ) = -\text{SIN}$$

$$\text{COS} = \frac{x}{r},\ \text{COS}(-\ ) = \frac{x}{r} \qquad \therefore \text{COS}(-\ ) = \text{COS}$$

$$\text{TAN} = \frac{y}{x},\ \text{TAN}(-\ ) = \frac{-y}{x} = -\left(\frac{y}{x}\right) \therefore \text{TAN}(-\ ) = -\text{TAN}$$

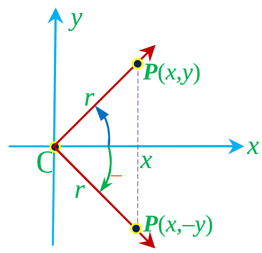


Figure 5.29

5.1.2 Values of Trigonometric Functions for Related Angles

The signs of sine, cosine and tangent functions

IN THIS SUB-SECTION YOU WILL CONSIDER WHETHER THE SIGN OF EACH OF THE TRIGONOMETRIC FUNCTIONS OF AN ANGLE IS POSITIVE OR NEGATIVE.

THE SIGN (WHETHER SIN, COS AND TAN ARE POSITIVE OR NEGATIVE) DEPENDS ON THE QUADRA TO WHICH BELONGS.

EXAMPLE 1 CONSIDER AN ANGLE IN THE FIRST AND SECOND QUADRANTS.

IF IS A FIRST QUADRANT ANGLE, THEN THE SIGN OF

SIN $= \frac{opp}{hyp} = \frac{y}{r}$ IS POSITIVE

COS $= \frac{adj}{hyp} = \frac{x}{r}$ IS POSITIVE

TAN $= \frac{opp}{adj} = \frac{y}{x}$ IS POSITIVE

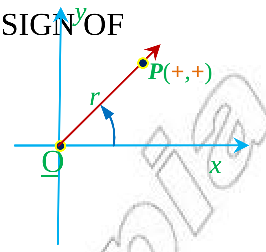


Figure 5.30

IF IS A SECOND QUADRANT ANGLE THEN, THE SIGN OF

SIN $= \frac{opp}{hyp} = \frac{y}{r}$ IS POSITIVE

COS $= \frac{adj}{hyp} = \frac{x}{r}$ IS NEGATIVE SINCE IS NEGATIVE

TAN $= \frac{opp}{adj} = \frac{y}{x}$ IS NEGATIVE

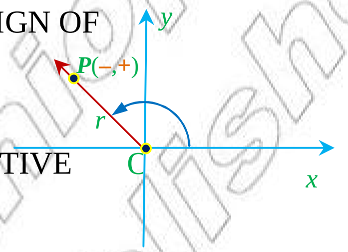


Figure 5.31

ACTIVITY 5.2

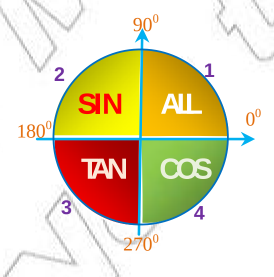
1 DETERMINE WHETHER THE SIGNS OF SIN , COS AND TAN ARE POSITIVE OR NEGATIVE:

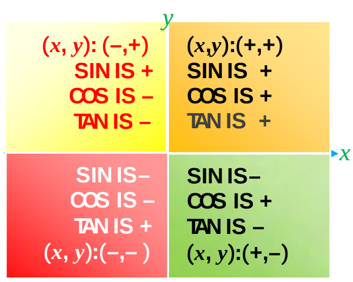
A IF IS A THIRD QUADRANT ANGLE **B** IF IS A FOURTH QUADRANT ANGLE

2 DECIDE WHETHER THE THREE TRIGONOMETRIC FUNCTIONS ARE POSITIVE OR NEGA COMPLETE THE FOLLOWING TABLE:

	has terminal side in quadrant			
	I	**II**	**III**	**IV**
sin	+			–
cos		–		
tan			+	

IN GENERAL, THE SIGNS OF THE SINE, COSINE AND TANGENT FUNCTIONS IN ALL OF THE CAN BE SUMMARIZED AS BELOW:





- IN THE FIRST QUADRANT all the three TRIGONOMETRIC FUNCTIONS ARE POSITIVE.
- IN THE SECOND QUADRANT ONLY sine IS POSITIVE.
- IN THE THIRD QUADRANT ONLY tangent IS POSITIVE.
- IN THE FOURTH QUADRANT ONLY cosine IS POSITIVE.

Do you want an easy way to remember this? KEEP IN MIND THE FOLLOWING STATEMENT:

All Students Take Chemistry

TAKING THE FIRST LETTER OF EACH WORD WE HAVE

All	All are positive
Students	Sine is positive
Take	Tangent is positive
Chemistry	Cosine is positive

EXAMPLE 2 DETERMINE THE SIGN OF:

A SIN 195° **B** TAN 336° **C** COS 895°

SOLUTION:

A OBSERVE THAT 180° < 195° < 270°. SO ANGLE 195° IS A THIRD QUADRANT ANGLE. IN THE THIRD QUADRANT THE SINE FUNCTION IS NEGATIVE.

∴ SIN 195° IS NEGATIVE

B SINCE 270° < 336° < 360°, THE ANGLE WHOSE MEASURE IS 336° IS A FOURTH QUADRANT ANGLE. IN THE FOURTH QUADRANT THE TANGENT FUNCTION IS N

HENCE TAN 336° IS NEGATIVE.

C SINCE 2(360)° < 895° < 2(360)° + 180°, THE ANGLE WHOSE MEASURE IS 895° IS A SECOND QUADRANT ANGLE. IN THE SECOND QUADRANT THE COSINE FUNCTION IS NEGATIVE.

HENCE, COS 895° IS NEGATIVE.

Group Work 5.3

1 DISCUSS AND ANSWER EACH OF THE FOLLOWING:

A IF TAN > 0 AND COS < 0, THEN IS IN QUADRANT______.

B IF SIN > 0 AND COS < 0, THEN IS IN QUADRANT______.

C IF COS > 0 AND TAN < 0, THEN IS IN QUADRANT______.

D IF SIN < 0 AND TAN > 0, THEN IS IN QUADRANT______.

2 DETERMINE THE SIGN OF:

A COS 267° **B** TAN (– 280°) **C** SIN (– 815°)

3 DETERMINE THE SIGNS OF SIN, COS AND TAN, IF IS AN ANGLE IN STANDARD POSITION AND P (2,–5) IS A POINT ON ITS TERMINAL SIDE.

Complementary angles

ANY TWO ANGLES ARE SAID TO BE **COMPLEMENTARY**, IF THE SUM OF THEIR MEASURES IS EQUAL TO 90^0.

EXAMPLE 3 ANGLE WITH MEASURES 30^O AND 60^O, 20^O AND 70^O, 40^O AND 50^O, 45^O AND 45^O, 10^O AND 80^O ARE EXAMPLES OF COMPLEMENTARY ANGLES.

ACTIVITY 5.3

1 REFERRING TO FIGURE 5.32,

A FIND SIN 30^0, COS 30^0, TAN 30^0, SIN 60^0, COS 60^0, TAN 60^0

B **I** COMPARE THE RESULTS OF SIN 30^0 AND COS 60^0.

II COMPARE THE RESULTS OF SIN 60^0 AND COS 30^0.

III COMPARE THE RESULTS OF TAN 30^0 AND TAN 60^0.

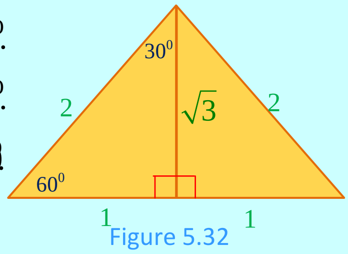


Figure 5.32

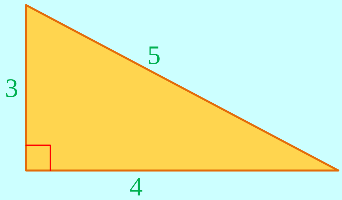
2 REFER TO FIGURE 5.33 ON THE RIGHT AND FIND

A SIN , COS , TAN, SIN , COS AND TAN.

B **I** COMPARE THE RESULTS OF SIN AND COS.

II COMPARE THE RESULTS OF SIN AND COS.

III COMPARE THE RESULTS OF TAN AND TAN.



C WHAT DO YOU CONCLUDE FROM YOUR FINDINGS?

Figure 5.33

FROM ACTIVITY 5.3, THE FOLLOWING RELATIONSHIPS CAN BE CONCLUDED:

IF AND ARE COMPLEMENTARY ANGLES, THAT IS,

(+ $= 90^0$) (*See* FIGURE 5.34), THEN WE HAVE,

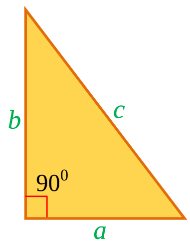


Figure 5.34

SIN $= \frac{a}{c}$ COS $= \frac{a}{c}$ TAN $= \frac{b}{a}$

SIN $= \frac{b}{c}$ COS $= \frac{b}{c}$ TAN $= \frac{a}{b} = \frac{1}{\left(\frac{b}{a}\right)}$

HENCE, FOR COMPLEMENTARY ANGLES AND,

SIN = COS , COS = SIN AND TAN $= \frac{1}{\text{TAN}}$.

Exercise 5.5

ANSWER EACH OF THE FOLLOWING QUESTIONS:

A IF $\sin 31^{\circ} = 0.5150$, THEN WHAT IS $\cos 59^{\circ}$?

B IF SIN $= \frac{3}{5}$, THEN WHAT IS COS $(90^{\circ} - \ )$?

C IF COS $= \frac{4}{5}$, THEN WHAT IS SIN $(90^{\circ} - \ )$?

D IF SIN $= k$, THEN WHAT IS COS $(90^{\circ} - \ )$?

E IF COS $= r$, THEN WHAT IS SIN $(90^{\circ} - \ )$?

F IF TAN $\beta = \frac{m}{n}$, THEN WHAT IS $\frac{1}{\tan(90^{\circ} - \ )}$?

Reference angle($_R$)

IF IS AN ANGLE IN STANDARD POSITION WHOSE TERMINAL SIDE DOES NOT LIE ON EITHER COORDINATE AXIS, THEN A **reference angle** $_R$ FOR IS THE **acute angle** FORMED BY THE TERMINAL SIDE OF AND THE x-AXIS AS SHOWN IN THE FOLLOWING FIGURES:

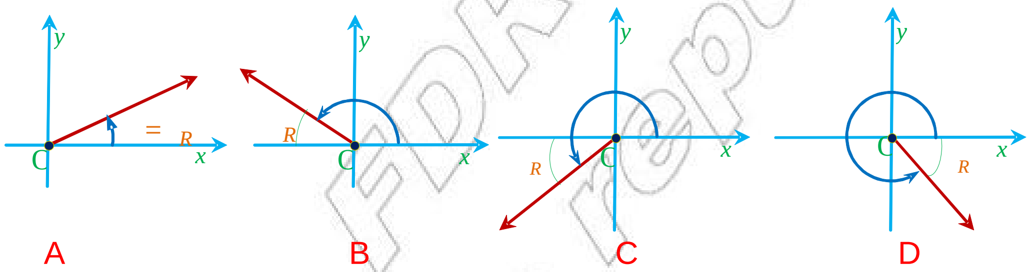


Figure 5.35

EXAMPLE 4 FIND THE REFERENCE ANGLE $_R$ FOR IF:

A $= 110°$ **B** $= 212°$ **C** $= 280°$

SOLUTION:

A SINCE $= 110°$ IS A SECOND QUADRANT ANGLE,

$_R = 180 - 110° = 70°$

B SINCE $= 212°$ IS A THIRD QUADRANT ANGLE,

$_R = 212° - 180° = 32°$

C SINCE $= 280°$ IS A FOURTH QUADRANT ANGLE,

$_R = 360° - 280° = 80°$

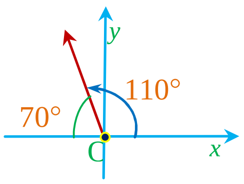


Figure 5.36

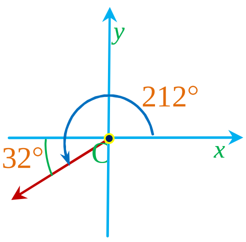


Figure 5.37

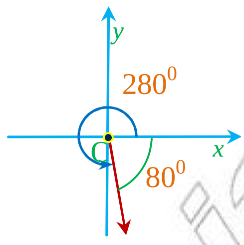


Figure 5.38

Exercise 5.6

FIND THE REFERENCE ANGLE θ_R FOR θ:

a $\theta = 150°$ **b** $\theta = 170°$ **c** $\theta = 240°$ **d** $\theta = 320°$

e $\theta = 99°$ **f** $\theta = 225°$ **g** $\theta = 315°$ **h** $\theta = 840°$

Values of the trigonometric functions of θ and its reference angle θ_R

LET US CONSIDER A SECOND QUADRANT ANGLE θ IN STANDARD POSITION AS SHOWN IN THE FIGURE 5.39, AND LET $P(-x, y)$ BE A POINT ON ITS TERMINAL SIDE. USING THE y-AXIS AS THE AXIS OF SYMMETRY, REFLECT P THROUGH THE y-AXIS. THIS WILL GIVE YOU ANOTHER POINT $P'(x, y)$ WHICH IS THE IMAGE OF P ON THE TERMINAL SIDE OF θ_R.

THIS IMPLIES THAT $OP = OP'$, THAT IS $OP = OP' = \sqrt{x^2 + y^2} = r$

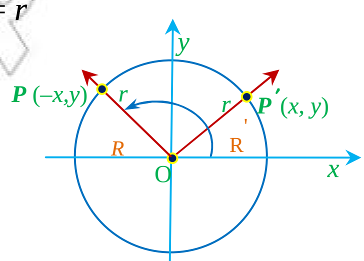


Figure 5.39

HENCE, $\sin\theta = \frac{y}{r}$, $\sin\theta_R = \frac{y}{r}$ $\Rightarrow$ $\sin\theta = \sin\theta_R$

$\cos\theta = \frac{-x}{r}$, $\cos\theta_R = \frac{x}{r}$ $\Rightarrow$ $\cos\theta = -\cos\theta_R$

$\tan\theta = \frac{y}{-x} = -\frac{y}{x}$, $\tan\theta_R = \frac{y}{x}$ $\Rightarrow$ $\tan\theta = -\tan\theta_R$

THE VALUES OF THE TRIGONOMETRIC FUNCTION OF θ AND THE VALUES OF THE CORRESPONDING TRIGONOMETRIC FUNCTIONS OF θ_R ARE THE SAME IN ABSOLUTE VALUE BUT THEY MAY DIFFER IN SIGN

EXAMPLE 5 EXPRESS THE SINE, COSINE AND TANGENT FUNCTIONS OF 160^O IN TERMS OF ITS REFERENCE ANGLE.

SOLUTION: ***Remember that*** AN ANGLE WITH MEASURE 160^O IS A SECOND QUADRANT ANGLE .

IN QUADRANT II, ONLY SINE IS POSITIVE.

THE REFERENCE ANGLE $\theta_R = 180^O - 160^O = 20^O$

THEREFORE, $\sin 160^O = \sin 20^O$, $\cos 160^O = -\cos 20^O$ AND $\tan 160^O = -\tan 20^O$

Supplementary angles

TWO ANGLES ARE SAID TO BE **supplementary**, IF THE SUM OF THEIR MEASURES IS EQUAL TO 180°.

EXAMPLE 6 PAIRS OF ANGLES WITH MEASURES OF 30° AND 150°, 120° AND 60°, 45° AND 135°, 75° AND 105°, 10° AND 170° ARE EXAMPLES OF SUPPLEMENTARY ANGLES.

EXAMPLE 7 FIND THE VALUES OF SIN 150°, COS 150° AND TAN 150°

SOLUTION: THE REFERENCE ANGLE $\theta_R = 180^{\circ} - 150^{\circ} = 30^{\circ}$

THEREFORE, $\sin 150^{\circ} = \sin 30^{\circ} = \frac{1}{2}$, $\cos 150^{\circ} = -\cos 30^{\circ} = -\frac{\sqrt{3}}{2}$

AND $\tan 150^{\circ} = -\tan 30^{\circ} = -\frac{\sqrt{3}}{3}$.

EXAMPLE 8 FIND THE VALUES OF SIN 240°, COS 240° AND TAN 240°

SOLUTION: THE REFERENCE ANGLE $\theta_R = 240^{0} - 180^{0} = 60^{0}$

$\sin 240^{\circ} = -\sin 60^{\circ} = -\frac{\sqrt{3}}{2}$, $\cos 240^{\circ} = -\cos 60^{\circ} = -\frac{1}{2}$ AND

$\tan 240^{\circ} = \tan 60^{\circ} = \sqrt{3}$.

… ***remember that*** *in quadrant III only tangent is positive.*

IN GENERAL,

IF θ IS A SECOND QUADRANT ANGLE, THEN ITS REFERENCE ANGLE WILL BE $(180^{\circ} - \theta)$. HENCE,

$\sin\theta = \sin(180^{\circ} - \theta)$ $\cos\theta = -\cos(180^{\circ} - \theta)$ $\tan\theta = -\tan(180^{\circ} - \theta)$

IF θ IS A THIRD QUADRANT ANGLE, ITS REFERENCE ANGLE WILL BE $\theta - 180^{\circ}$.

HENCE, $\sin\theta = -\sin(\theta - 180^{\circ})$ $\cos\theta = -\cos(\theta - 180^{\circ})$ AND $\tan\theta = \tan(\theta - 180^{\circ})$.

Exercise 5.7

1 EXPRESS THE SINE, COSINE AND TANGENT FUNCTIONS OF THE FOLLOWING ANGLE MEASURES IN TERMS OF THEIR REFERENCE ANGLE:

A 105° **B** 175° **C** 220°

D -260° **E** -300° **F** 380°

2 FIND THE VALUES OF:

A SIN 135°, COS 135° AND TAN 135° **B** COS 143°, IF COS $37^{\circ} = 0.7986$

C TAN 138° IF TAN $42^{\circ} = 0.9004$ **D** SIN 115°, IF SIN $65^{\circ} = 0.9063$

E TAN 159° IF TAN $21^{\circ} = 0.3839$ **F** COS 24°, IF COS $156^{\circ} = -0.9135$

Co–terminal angles

Co–terminal angles ARE ANGLES IN STANDARD POSITION THAT HAVE A COMMON TERMINAL SIDE.

EXAMPLE 9

A THE THREE ANGLES WITH MEASURES –30°, 330° AND 390° ARE CO-TERMINAL ANGLES. (*See* FIGURE 5.40)

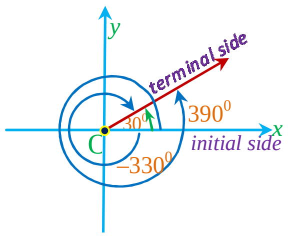


Figure 5.40

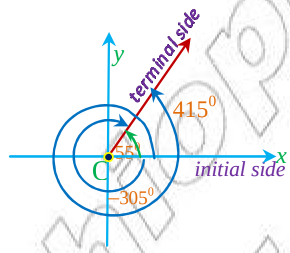


Figure 5.41

B THE THREE ANGLES WITH MEASURES 55°, –305° AND 415° ARE ALSO CO-TERMINAL. (*See* FIGURE 5.41)

ACTIVITY 5.4

1 WITH THE HELP OF THE FOLLOWING TABLE FIND ANGLES WHICH ARE CO-TERMINAL WITH 60°.

Angles which are co–terminal with 60°	
$60^\circ + 1(360^\circ) = 420^\circ$	$60^\circ - 1(360^\circ) = -300^\circ$
$60^\circ + 2(360^\circ) = 780^\circ$	$60^\circ - 2(360^\circ) = -660^\circ$
________________	________________
________________	________________
________________	________________
$60^\circ + 6(360^\circ) = 2220^\circ$	$60^\circ - 6(360^\circ) = -2100^\circ$
.	.
.	.
.	.

2 GIVE A FORMULA TO FIND ALL ANGLES WHICH ARE CO-TERMINAL WITH 60°.

GIVEN AN ANGLE , ALL ANGLES WHICH ARE CO–TERMINAL WITH ARE GIVEN BY THE FORMULA

$\pm n\,(360^\circ)$, WHERE $n = 1, 2, 3, \ldots$

EXAMPLE 10 FIND A POSITIVE AND A NEGATIVE ANGLE CO-TERMINAL WITH 75°.

SOLUTION: TO FIND A POSITIVE AND A NEGATIVE ANGLE CO-TERMINAL WITH A GIVEN ANGLE, YOU CAN ADD OR SUBTRACT 360°. HENCE, 75° – 360° = –285°; 75° + 360° = 435°.

THEREFORE, –285° AND 435° ARE CO-TERMINAL WITH 75°.

THERE ARE AN INFINITE NUMBER OF OTHER ANGLES CO-TERMINAL WITH 75°. THEY
BY 75°± n (360°), n = 1, 2, 3, . . .

Exercise 5.8

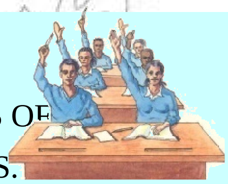
FIND ANY TWO CO-TERMINAL ANGLES (ONE OF THEM POSITIVE AND THE OTHER NEGAT
OF THE FOLLOWING ANGLE MEASURES:

A 70^{O} **B** 110^{O} **C** 220^{O} **D** 270^{O}

E -90^{O} **F** -37^{O} **G** -60^{O} **H** -70^{O}

Trigonometric values of co–terminal angles

ACTIVITY 5.5



CONSIDER FIGURE 5.42 AND FIND THE TRIGONOMETRIC VALUES OF AND .
P (x, y) IS A POINT ON THE TERMINAL SIDE OF BOTH ANGLES.
ANSWER EACH OF THE FOLLOWING QUESTIONS:

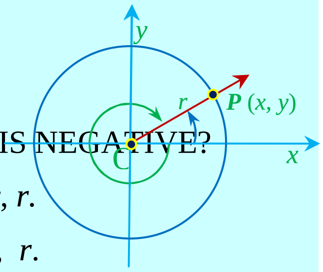


Figure 5.42

A ARE AND CO-TERMINAL ANGLES? WHY?

B WHICH ANGLE IS POSITIVE? WHICH ANGLE IS NEGATIVE?

C FIND THE VALUES OF SIN , COS , TAN IN TERMS OF x, y, r.

D FIND THE VALUES OF SIN , COS , TAN IN TERMS OF x, y, r.

E IS SIN = SIN ? IS COS = COS ? IS TAN = TAN ?

F WHAT CAN YOU CONCLUDE ABOUT THE TRIGONOMETRIC VALUES OF CO-TERMINAL ANGLES?

CO-TERMINAL ANGLES HAVE THE SAME TRIGONOMETRIC VALUES.

EXAMPLE 11 FIND THE TRIGONOMETRIC VALUES OF

A -330^{O} AND 30^{O} **B** 120^{O} AND -240^{O}

SOLUTION:

A OBSERVE THAT BOTH ANGLES ARE CO-TERMINAL. THEIR TERMINAL SIDE LIES IN THE FIRST QUADRANT (SEE FIGURE 5.43).

$-330^{O} = 30^{O} - 1(360^{O})$. THIS GIVES US:

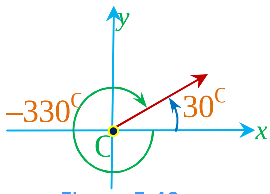


Figure 5.43

$\sin 30^{O} = \sin(-330^{O}) = \frac{1}{2}$

$\cos 30^{O} = \cos(-330^{O}) = \frac{\sqrt{3}}{2}$

$\tan 30^{O} = \tan(-330^{O}) = \frac{\sqrt{3}}{3}$

B BOTH 120^{O} AND -240^{O} ANGLES ARE CO-TERMINAL. THEIR TERMINAL SIDE LIES IN THE SECOND QUADRANT. (*See* FIGURE 5.44)

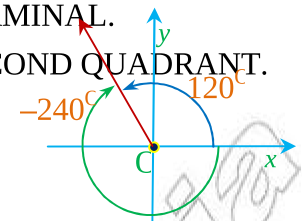


Figure 5.44

$-240^{o} = 120^{o} - 360^{o}$. THUS,

SIN 120^{O} = SIN (-240^{O}) = SIN 60^{O} = $\dfrac{\sqrt{3}}{2}$

… *a* 60^{0} *angle is the reference angle for a* 120^{0} *angle*

COS 120^{O} = COS (-240^{O}) = – COS 60^{O} = $-\dfrac{\sqrt{3}}{2}$

… *cosine is negative in quadrant II*

TAN 120^{O} = TAN (-240^{O}) = – TAN 60^{O} = $-\sqrt{3}$

… *tangent is also negative in quadrant II*

Angles larger than 360°

CONSIDER THE 780^{O} ANGLE

$780^{O} = 360^{O} + 360^{O} + 60^{O} = 2(360^{O}) + 60^{O}$

… *a* 60^{o} *angle is the co-terminal acute angle for a* 780^{O} *angle*

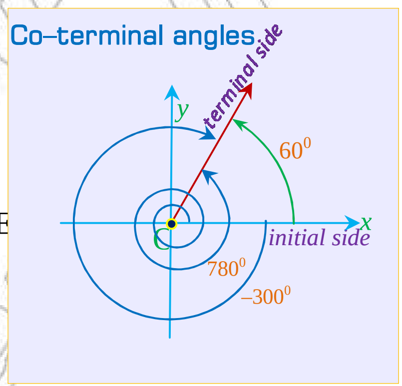


Figure 5.45

SINCE AN ANGLE AND ITS CO–TERMINAL HAVE THE SAME TRIGONOMETRIC VALUE,

SIN 780^{O} = SIN 60^{O} = $\dfrac{\sqrt{3}}{2}$, COS 780^{O} = COS 60^{O} = $\dfrac{1}{2}$

AND TAN 780^{O} = TAN 60^{O} = $\sqrt{3}$.

(***Remember that*** *since 780º is the measure of a first quadrant angle, all three of the functions are positive.*)

EXAMPLE 12 FIND THE TRIGONOMETRIC VALUES OF 945^{O}

SOLUTION: $945^{O} = 360^{O} + 360^{O} + 225^{O} = 2(360^{O}) + 225^{O}$

THIS MEANS 945^{O} AND 225^{O} ARE MEASURES OF CO-TERMINAL 3^{RD} QUADRANT ANGLES.

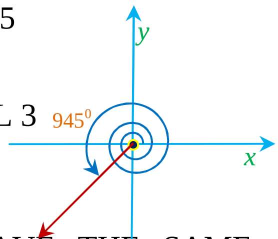


Figure 5.46

THE REFERENCE ANGLE FOR 225^{O} IS $\theta_R = 225^{O} - 180^{O} = 45^{O}$.

SINCE AN ANGLE AND ITS CO-TERMINAL HAVE THE SAME TRIGONOMETRIC VALUE, IT FOLLOWS THAT

SIN 945^{O} = SIN 225^{O} = – SIN 45^{O} = $-\dfrac{\sqrt{2}}{2}$ … *sine is negative in quadrant III*

COS 945^{O} = COS 225^{O} = – COS 45^{O} = $-\dfrac{\sqrt{2}}{2}$ … *cosine is negative in quadrant III*

TAN 945^{O} = TAN 225^{O} = TAN 45^{O} = 1 … *tangent is positive in quadrant III*

Exercise 5.9

1 FIND THE VALUE OF EACH OF THE FOLLOWING:

- **A** $\sin 390^\circ$, $\cos 390^\circ$, $\tan 390^\circ$
- **B** $\sin(-405^\circ)$, $\cos(-405^\circ)$, $\tan(-405^\circ)$
- **C** $\sin(-690^\circ)$, $\cos(-690^\circ)$, $\tan(-690^\circ)$
- **D** $\sin 1395^\circ$, $\cos 1395^\circ$, $\tan 1395^\circ$

2 EXPRESS EACH OF THE FOLLOWING AS A TRIGONOMETRIC FUNCTION OF A POSITIVE ACUTE ANGLE:

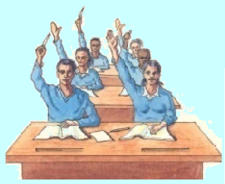
A	$\sin 130^\circ$	**B**	$\sin 200^\circ$	**C**	$\cos 165^\circ$	**D**	$\cos 310^\circ$
E	$\tan 325^\circ$	**F**	$\sin(-100^\circ)$	**G**	$\cos(-305^\circ)$	**H**	$\tan 415^\circ$
I	$\sin 1340^\circ$	**J**	$\tan 1125^\circ$	**K**	$\sin(-330^\circ)$	**L**	$\cos 1400^\circ$

5.1.3 Graphs of the Sine, Cosine and Tangent Functions

IN THIS SECTION, YOU WILL DRAW AND DISCUSS SOME PROPERTIES OF THE GRAPHS OF THE THREE TRIGONOMETRIC FUNCTIONS: SINE, COSINE AND TANGENT.

Graph of the sine function

ACTIVITY 5.6



1 COMPLETE THE FOLLOWING TABLE OF VALUES FOR $y = \sin\theta$.

θ in deg	–360	–330	–270	–240	–180	–120	–90
$y = \sin\theta$							

θ in deg	0	90	120	180	240	270	330	360
$y = \sin\theta$								

2 MARK THE VALUES OF θ ON THE HORIZONTAL AXIS AND THE VALUES OF y ON THE VERTICAL AXIS AND PLOT THE POINTS YOU FIND.

3 CONNECT THESE POINTS USING A SMOOTH CURVE TO DRAW THE GRAPH OF $y = \sin\theta$.

4 WHAT ARE THE DOMAIN AND THE RANGE OF $y = \sin\theta$?

EXAMPLE 1 DRAW THE GRAPH OF $y = \sin\theta$, WHERE $-360^\circ \le \theta \le 360^\circ$.

SOLUTION: TO DETERMINE THE GRAPH OF $y = \sin\theta$, WE CONSTRUCT A TABLE OF VALUES FOR $y = \sin\theta$, WHERE $-360^\circ \le \theta \le 360^\circ$ (WHICH IS THE SAME AS $-2\pi \le \theta \le 2\pi$ IN *radians*.)

THE TABLES BELOW SHOW SOME OF THE VALUES OF $y = \sin\theta$ IN THE GIVEN INTERVAL.

in *deg*	–360	–330	–300	–270	–240	–210	–180	–150	–120	– 90	– 60	–30
in *rad*	-2	$-\frac{11}{6}$	$-\frac{5}{3}$	$-\frac{3}{2}$	$-\frac{4}{3}$	$-\frac{7}{6}$	$-\pi$	$-\frac{5}{6}$	$-\frac{2}{3}$	$-\frac{}{2}$	$-\frac{}{3}$	$-\frac{}{6}$
y = sin	0	0.5	0.87	1	0.87	0.5	0	– 0.5	– 0.87	– 1	– 0.87	– 0.5

in deg	0	30	60	90	120	150	180	210	240	270	300	330	360
in rad	0	$\frac{}{6}$	$\frac{}{3}$	$\frac{}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	π	$\frac{7}{6}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{11}{6}$	2
y = sin	0	0.5	0.87	1	0.87	0.5	0	– 0.5	– 0.87	–1	– 0.87	– 0.5	0

TO DRAW THE GRAPH WE MARK THE VALUES OF θ ON THE HORIZONTAL AXIS AND THE VALUES OF *y* ON THE VERTICAL AXIS. THEN WE PLOT THE POINTS AND CONNECT THEM USING A SMOOTH

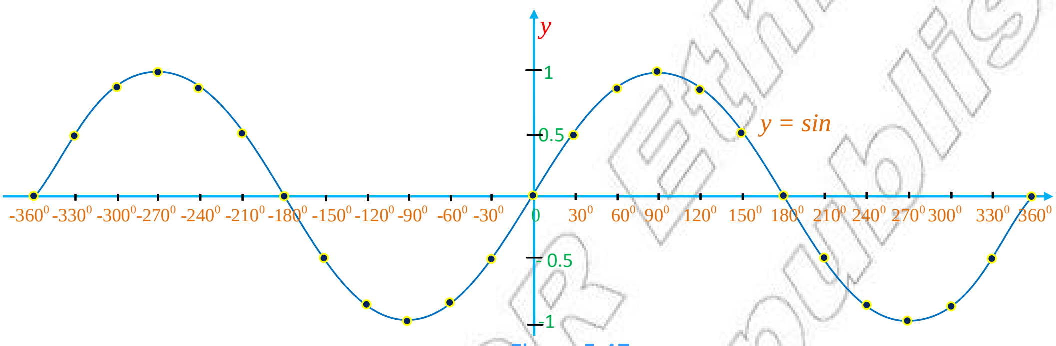


Figure 5.47

AFTER A COMPLETE REVOLUTION (EVERY 360° OR 2π RADIAN) THE VALUES OF THE SINE FUNCTION REPEAT THEMSELVES. THIS MEANS

SIN θ = SIN (θ ± 360°) = SIN (θ ± 2(360°)) = SIN (θ ± 3(360°)), ETC.

SIN 90° = SIN (90° ± 360°) = SIN (90° ± 2(360°)) = SIN (90° ± 3(360°)), ETC.

SIN 180° = SIN (180° ± 360°) = SIN (180° ± 2(360°)) = SIN (180° ± 3(360°)), ETC.

IN GENERAL, SIN θ = SIN (θ ± *n* (360°)) WHERE *n* IS AN INTEGER.

A FUNCTION THAT REPEATS ITS VALUES AT REGULAR INTERVALS IS CALLED A **periodic function**.

THE SINE FUNCTION REPEATS AFTER EVERY 360° OR 2π RADIANS.

THEREFORE, 360° OR 2π IS CALLED THE **period** OF THE SINE FUNCTION.

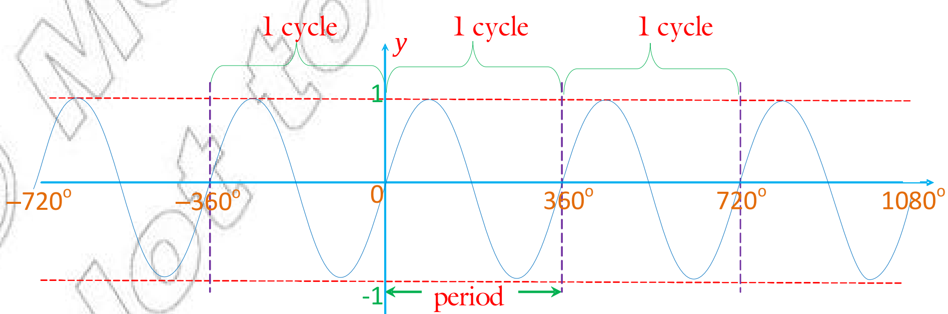


Figure 5.48 Graph of $y = \sin\theta$ for $-720^\circ \le \theta \le 1080^\circ$

Domain and range

FOR ANY ANGLE θ TAKEN ON THE UNIT CIRCLE, THERE IS SOME POINT P(x, y) ON ITS TERMINAL SIDE. SINCE SIN $\theta = \frac{y}{1} = y$, THE FUNCTION $y =$ SIN θ IS DEFINED FOR ANY ANGLE θ TAKEN ON THE UNIT CIRCLE.

THEREFORE, THE DOMAIN OF THE SINE FUNCTION IS THE SET OF ALL REAL NUMBERS.

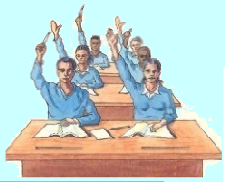
ALSO, NOTE FROM THE GRAPH THAT THE VALUE OF Y IS NEVER LESS THAN –1 OR GREAT

Note: THE DOMAIN OF THE SINE FUNCTION IS THE SET OF ALL REAL NUMBERS R

THE RANGE OF THE SINE FUNCTION IS $\{y \mid -1 \le y \le 1\}$

Graph of the cosine function

ACTIVITY 5.7



1 COMPLETE THE FOLLOWING TABLES OF VALUES FOR $y =$ COS θ.

θ in deg	– 360	– 300	– 270	– 240	– 180	– 120	– 90	– 60
y = cos θ								

θ in deg	0	60	90	120	180	240	270	300	360
y = cos θ									

2 SKETCH THE GRAPH OF $y =$ COS θ.

3 WHAT ARE THE DOMAIN AND THE RANGE OF $y =$ COS θ?

4 WHAT IS THE PERIOD OF THE COSINE FUNCTION?

FROM ACTIVITY 5.7, YOU CAN SEE THAT $y =$ COS θ IS NEVER LESS THAN –1 OR GREATER THAN +1.

JUST LIKE THE SINE FUNCTION, THE COSINE FUNCTION IS PERIODIC. IT REPEATS EVERY 360° OR 2π RADIANS.

THEREFORE, 360° OR 2π IS CALLED THE period OF THE COSINE FUNCTION.

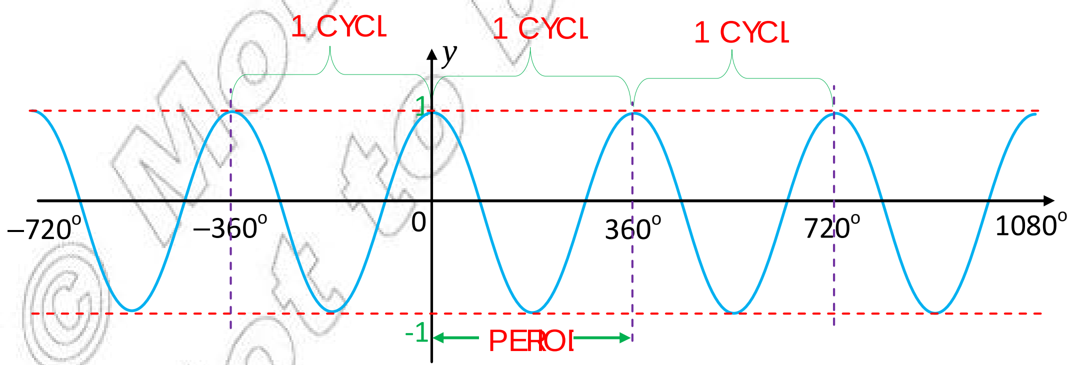


Figure 5.49 Graph of $y =$ COS θ for $-720° \le \theta \le 1080°$

Note: The domain of the cosine function is the set of all real numbers, R.

The range of the cosine function is $\{y \mid -1 \le y \le 1\}$.

Figure 5.50 represents the sine and cosine functions on the same coordinate system.

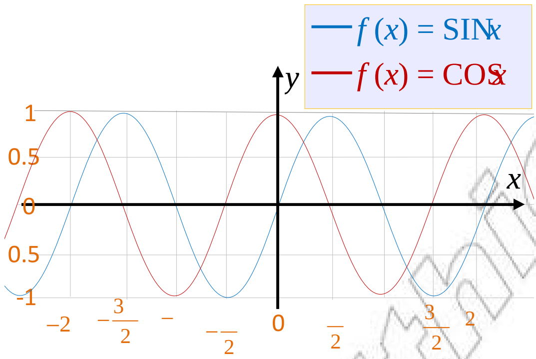


Figure 5.50

From this diagram you can see that both the sine and cosine curves have the same shape.

The curves "follow" each other, always $\frac{\pi}{2}$ radians (90°) apart.

Graph of the tangent function

ACTIVITY 5.8

1 Complete the following tables of values for $y = \tan x$.

x in deg	–360	–315	–270	–225	–180	–135	–90	–45
$y = \tan x$								

x in deg	0	45	90	135	180	225	270	315	360
$y = \tan x$									

2 Use the table you constructed above to sketch the graph of $y = \tan x$.

3 For which values of x is $y = \tan x$ undefined?

4 What are the domain and the range of $y = \tan x$?

5 What is the period of the tangent function?

The Activity 5.8 you have done above gives you a hint on what the graph of $y = \tan x$ looks like. Next, you will see the graph in detail.

EXAMPLE 2 DRAW THE GRAPH OF y = TAN, WHERE –360° ≤ ≤ 360°.

SOLUTION: THE TABLES BELOW SHOW SOME OF THE VALUES OF y = TAN,

WHERE –2 ≤ ≤ 2

θ* in *deg	–360	–315	–270	–225	–180	–135	–90	–45	0
θ* in *rad	–2	$-\frac{7}{4}$	$-\frac{3}{2}$	$-\frac{5}{4}$	–	$-\frac{3}{4}$	$-\frac{}{2}$	$-\frac{}{4}$	0
y* = tan *θ	0	1	–	–1	0	1	–	–1	0

θ* in *deg	45	90	135	180	225	270	315	360
θ* in *rad	$\frac{}{4}$	$\frac{}{2}$	$\frac{3}{4}$		$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2
y* = tan *θ	1	–	–1	0	1	–	–1	0

Remember that IF IS IN A STANDARD POSITION AND $P(x, y)$ IS A POINT WHERE THE TERMINAL SIDE OF INTERSECTS THE UNIT CIRCLE, THEN TAN $= \frac{y}{x}$. HOWEVER, $\frac{y}{x}$ IS NOT DEFINED IF $x = 0$.

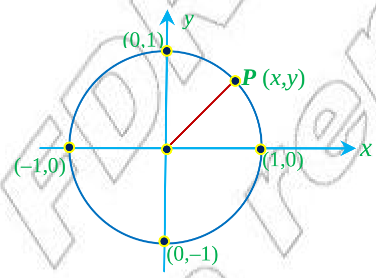


Figure 5.51

SO TAN IS NOT DEFINED IF

= 90°, = 90° ± 180°, = 90° ± 2(180°), = 90° ± 3(180°), ETC.

IN GENERAL, TAN IS UNDEFINED IF = 90° ± n (180°) OR IF $= \frac{}{2} + n$, WHERE n IS AN INTEGER.

THE GRAPH OF y = TAN DOES NOT CROSS THE VERTICAL LINES $= \frac{}{2} + n$, n IS INTEGER.

MOREOVER, IF WE CLOSELY INVESTIGATE THE BEHAVIOUR OF TAN AS INCREASES FROM $-\frac{}{2}$ TO $\frac{}{2}$, WE CAN SEE THAT TAN INCREASES FROM NEGATIVE INFINITY TO POSITIVE INFINITY (FROM –∞ TO ∞). A SKETCH OF THE GRAPH OF y = TAN FOR $-\frac{}{2} < < \frac{}{2}$, IS SHOWN IN FIGURE 5.52.

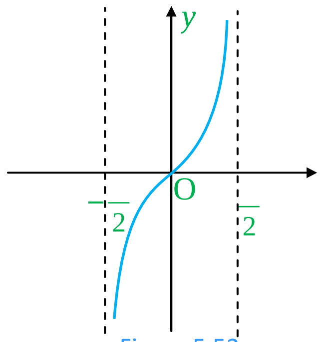


Figure 5.52

FROM THE GRAPH WE SEE THAT THE TANGENT FUNCTION REPEATS ITSELF EVERY 180° OR π RADIANS.

THEREFORE, 180° OR π IS THE PERIOD FOR THE TANGENT FUNCTION.

SINCE TAN IS PERIODIC WITH PERIOD π, WE CAN EXTEND THE ABOVE GRAPH FOR AS MANY REPETITIONS (CYCLES) AS WE WANT.

FOR EXAMPLE, THE GRAPH OF $y = \tan\theta$ FOR $-2\pi \le \theta \le 2\pi$ IS SHOWN BELOW.

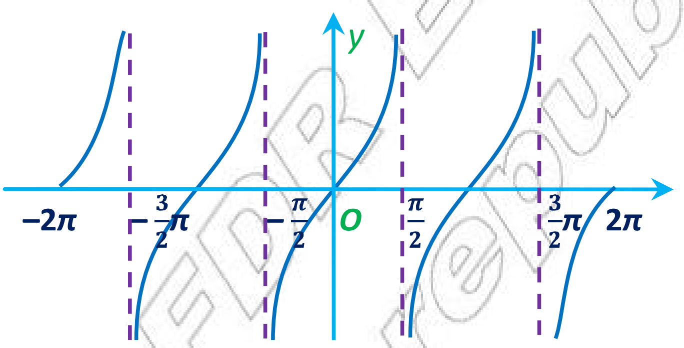


Figure 5.53

WHAT ARE THE DOMAIN AND THE RANGE OF $y = \tan\theta$?

FOR WHICH VALUES OF θ IS $y = \tan\theta$ NOT DEFINED?

USING A UNIT CIRCLE WE CAN SEE THAT $\tan\theta = \frac{y}{x}$ IS UNDEFINED WHENEVER THE x-COORDINATE ON THE UNIT CIRCLE IS 0.

THIS HAPPENS WHEN $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}$, ETC. THEREFORE THE DOMAIN OF THE TANGENT FUNCTION MUST EXCLUDE THESE ODD MULTIPLES OF $\frac{\pi}{2}$.

HENCE, THE DOMAIN OF THE TANGENT FUNCTION IS $\{\theta \mid \theta \ne n\frac{\pi}{2}$, WHERE n IS AN ODD INTEGER$\}$.

THE RANGE OF $y = \tan\theta$ IS THE SET OF REAL NUMBERS.

Group Work 5.4

1 USE THE GRAPH OF THE COSINE FUNCTION TO FIND ALL VALUES OF x FOR WHICH $\cos x = 0$.

2 FROM THE GRAPH OF $y = \sin x$, FIND THE VALUES OF x FOR WHICH $\sin x = -1$.

3 GRAPH THE SINE CURVE FOR THE INTERVAL $-540^\circ \le x \le 0^\circ$.

Exercise 5.10

1 REFER TO THE GRAPH OF $y = \sin x$ OR THE TABLE OF VALUES FOR $y = \sin x$ TO DETERMINE HOW THE SINE FUNCTION BEHAVES AS x INCREASES FROM 0° TO 360° AND ANSWER THE FOLLOWING:

- **A** AS x INCREASES FROM 0° TO 90°, $\sin x$ INCREASES FROM __0__ TO __1__.
- **B** AS x INCREASES FROM 90° TO 180°, $\sin x$ DECREASES FROM ______ TO _____.
- **C** AS x INCREASES FROM 180° TO 270°, $\sin x$ DECREASES FROM _____ TO _____.
- **D** AS x INCREASES FROM 270° TO 360°, $\sin x$ INCREASES FROM ______ TO ____.

2 REFER TO THE GRAPH OF $y = \cos x$ OR THE TABLE OF VALUES FOR $y = \cos x$ TO DETERMINE HOW THE COSINE FUNCTION BEHAVES AS x INCREASES FROM 0° TO 360° AND ANSWER THE FOLLOWING:

- **A** AS x INCREASES FROM 0° TO 90°, $\cos x$ DECREASES FROM __1__ TO __0__.
- **B** AS x INCREASES FROM 90° TO 180°, $\cos x$ DECREASES FROM ______ TO _____.
- **C** AS x INCREASES FROM 180° TO 270°, $\cos x$ INCREASES FROM _____ TO _____.
- **D** AS x INCREASES FROM 270° TO 360°, $\cos x$ INCREASES FROM ______ TO ____.

3 DETERMINE HOW THE TANGENT FUNCTION BEHAVES AS x INCREASES FROM 0° TO 360° AND ANSWER THE FOLLOWING:

- **A** AS x INCREASES FROM 0° TO 90°, $\tan x$ INCREASES FROM __0__ TO __POSITIVE INFINITY ($+\infty$)__.
- **B** AS x INCREASES FROM 90° TO 180°, $\tan x$ INCREASES FROM ___ TO ____.
- **C** AS x INCREASES FROM 180° TO 270°, $\tan x$ INCREASES FROM __ TO _____.
- **D** AS x INCREASES FROM 270° TO 360°, $\tan x$ ______ FROM $-\infty$ TO 0.

5.2 THE RECIPROCAL FUNCTIONS OF THE BASIC TRIGONOMETRIC FUNCTIONS

IN THIS SECTION, YOU WILL LEARN ABOUT THREE MORE TRIGONOMETRIC FUNCTIONS, WHICH ARE CALLED THE RECIPROCALS OF THE SINE, COSINE AND TANGENT FUNCTIONS, NAMED RESPECTIVELY cosecant, secant AND cotangent FUNCTIONS.

5.2.1 The Cosecant, Secant and Cotangent Functions

Definition 5.2

If θ is an angle in standard position and **P**(x, y) is a point on the terminal side of θ, different from the origin **O**(0, 0), and r is the distance of point **P** from the origin O, then

$$\csc\theta = \frac{HYP}{OPP} = \frac{r}{y}$$

$$\sec\theta = \frac{HYP}{ADJ} = \frac{r}{x}$$

$$\cot\theta = \frac{ADJ}{OPP} = \frac{x}{y}$$

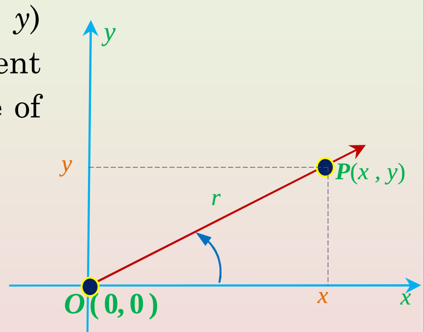


Figure 5.54

CSC, SEC AND COT ARE ABBREVIATIONS FOR COSECANT, SECANT AND COTANGENT RESPECTIVELY.

EXAMPLE 1 IF θ IS AN ANGLE IN STANDARD POSITION AND P(3, 4) IS A POINT ON THE TERMINAL SIDE OF θ, THEN EVALUATE THE COSECANT, SECANT AND COTANGENT FUNCTIONS.

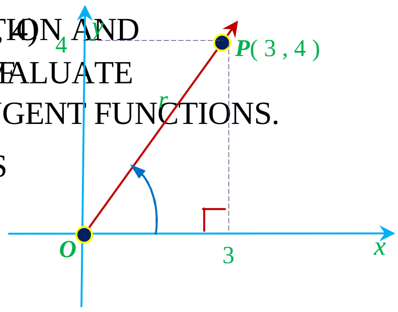


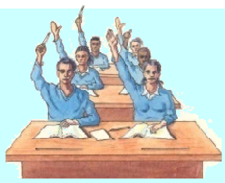
Figure 5.55

SOLUTION: THE DISTANCE $r = \sqrt{3^2+4^2} = \sqrt{25} = 5$ UNITS

SO, $\text{CSC}\,\theta = \frac{HYP}{OPP} = \frac{5}{4}$,

$\text{SEC}\,\theta = \frac{HYP}{ADJ} = \frac{5}{3}$ AND $\text{COT}\,\theta = \frac{ADJ}{OPP} = \frac{3}{4}$

ACTIVITY 5.9



REFERRING TO FIGURE 5.55 FIND:

1. SIN θ, COS θ AND TAN θ
2. COMPARE SIN θ WITH CSC θ, COS θ WITH SEC θ, TAN θ WITH COT θ
3. HOW DO THEY RELATE? ARE THEY EQUAL? ARE THEY OPPOSITE? ARE THEY RECIPROCALS?

FROM THE RESULTS OF ACTIVITY 5.9, YOU CAN CONCLUDE THE FOLLOWING:

$$\text{CSC} = \frac{r}{y} \qquad \text{WHEREAS} \qquad \text{SIN} = \frac{y}{r}$$

$$\text{SEC} = \frac{r}{x} \qquad \text{WHEREAS} \qquad \text{COS} = \frac{x}{r}$$

$$\text{COT} = \frac{x}{y} \qquad \text{WHEREAS} \qquad \text{TAN} = \frac{y}{x}$$

Have you noticed that one is the reciprocal of the other?

THAT IS,

$$\text{CSC} = \frac{r}{y} = \frac{1}{\frac{y}{r}} = \frac{1}{\text{SIN}},\ \text{SEC} = \frac{r}{x} = \frac{1}{\left(\frac{x}{r}\right)} = \frac{1}{\text{COS}} \ \text{AND}$$

$$\text{COT} = \frac{x}{y} = \frac{1}{\left(\frac{y}{x}\right)} = \frac{1}{\text{TAN}}$$

THEREFORE,

$$\text{CSC}\,\theta = \frac{1}{\text{SIN}},\ \text{SEC}\,\theta = \frac{1}{\text{COS}} \ \text{AND COT}\,\theta = \frac{1}{\text{TAN}}.$$

HENCE, CSC AND SIN ARE RECIPROCALS

SEC AND COS ARE RECIPROCALS

TAN AND COT ARE RECIPROCALS

EXAMPLE 2 IF = 30^{O}, THEN FIND CSC, SEC, COT.

SOLUTION:

$$\text{CSC} = \frac{1}{\text{SIN}} = \frac{1}{\left(\frac{1}{2}\right)} = 2 \qquad \ldots \textbf{\textit{remember that}} \sin 30^0 = \frac{1}{2} = 0.5$$

$$\text{SEC} = \frac{1}{\text{COS}} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \ldots \textbf{\textit{remember that}} \cos 30^0 = \frac{\sqrt{3}}{2}$$

$$\text{COT} = \frac{1}{\text{TAN}} = \frac{1}{\left(\frac{\sqrt{3}}{3}\right)} = \frac{3}{\sqrt{3}} = \sqrt{3} \qquad \ldots \textbf{\textit{remember that}} \tan 30^0 = \frac{\sqrt{3}}{3}$$

EXAMPLE 3 IF SIN IS 0.5, THEN CSC IS $\frac{1}{0.5} = 2$

IF COS IS – 0.1035, THEN SEC IS $\frac{1}{-0.1035} = -9.6618$

IF TAN IS $-\frac{1}{4}$, THEN COT IS $\frac{1}{\left(-\frac{1}{4}\right)} = -4$

EXAMPLE 4 USING A UNIT CIRCLE, FIND THE VALUES OF THE COSECANT, SECANT AND COTANGENT FUNCTIONS IF = 90^{O}, 180^{O}, 270^{O}.

SOLUTION: AS YOU CAN SEE IN THE ADJACENT FIGURE, THE TERMINAL SIDE OF THE ANGLE 90^{O} INTERSECTS THE UNIT CIRCLE AT (0 ,1)

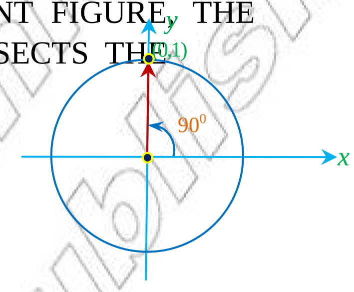


Figure 5.56

HENCE, $\csc 90^{O} = \frac{r}{y} = \frac{1}{1} = 1$

$\sec 90^{O} = \frac{r}{x} = \frac{1}{0}$ *is undefined*

$\cot 90^{O} = \frac{x}{y} = \frac{0}{1} = 0$

THE TERMINAL SIDE OF THE ANGLE 180^{O} INTERSECTS THE *unit circle* AT (–1, 0).

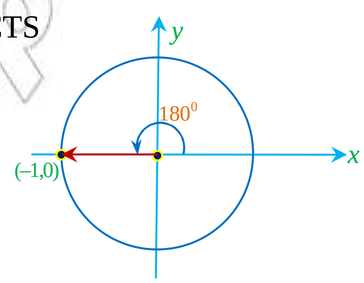


Figure 5.57

HENCE, $\csc 180^{O} = \frac{r}{y} = \frac{1}{0}$ *is undefined*

$\sec 180^{O} = \frac{r}{x} = \frac{1}{-1} = -1$

$\cot 180^{O} = \frac{x}{y} = \frac{-1}{0}$ *is undefined*

SIMILARLY THE TERMINAL SIDE OF THE ANGLE 270^{O} INTERSECTS THE *unit circle* AT (0 ,–1).

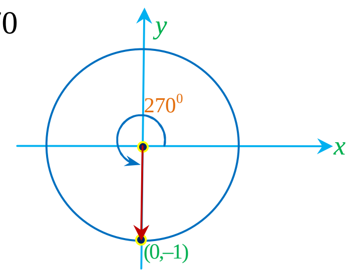


Figure 5.58

HENCE, $\csc 270^{O} = \frac{r}{y} = \frac{1}{-1} = -1$

$\sec 270^{O} = \frac{r}{x} = \frac{1}{0}$ *is undefined*

$\cot 270^{O} = \frac{x}{y} = \frac{0}{-1} = 0$

EXAMPLE 5 USING A UNIT CIRCLE, FIND THE VALUES OF THE COSECANT, SECANT AND COTANGENT FUNCTIONS IF = 360^{O}.

SOLUTION: THE TERMINAL SIDE OF AN ANGLE 360° INTERSECTS THE UNIT CIRCLE AT (1, 0).

HENCE, $\text{CSC } 360^{\circ} = \frac{r}{y} = \frac{1}{0}$ *is undefined*

$\text{SEC } 360^{\circ} = \frac{r}{x} = \frac{1}{1} = 1$

$\text{COT } 360^{\circ} = \frac{x}{y} = \frac{1}{0}$ *is undefined*

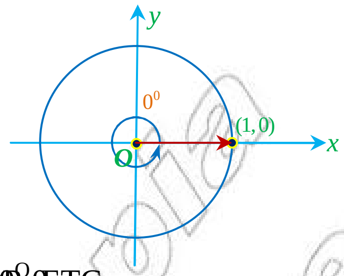


Figure 5.59

Remember that THESE RESULTS ARE ALSO TRUE FOR 0°, 720°, 1080°, ETC.

WHEN DO YOU THINK THE FUNCTIONS CSC, SEC AND COT ARE UNDEFINED?

FOR EXAMPLE, $\text{CSC } \theta = \frac{r}{y}$ IS UNDEFINED WHEN $y = 0$. THE VALUE OF y ON THE UNIT CIRCLE WILL BE 0 WHEN $\theta = 0^{\circ}, \pm 180^{\circ}, \pm 2(180^{\circ}), \pm 3(180^{\circ}), \pm 4(180^{\circ})$, ETC.

IN GENERAL, CSC θ IS UNDEFINED FOR $\theta = \pm n\,(180^{\circ})$, WHERE n IS AN INTEGER.

Group Work 5.5

1 DECIDE IF THE FOLLOWING TRIGONOMETRIC FUNC
POSITIVE OR NEGATIVE AND COMPLETE THE FOLLOW

	has terminal side in quadrant			
	I	II	III	IV
csc	+			
sec			–	
cot				–

2 COMPLETE THE FOLLOWING TABLE OF VALUES:

θ in deg	–360	–300	–270	–240	–180	–120	–90	–60	0
y = csc θ									
y = sec θ									

θ in deg	60	90	120	180	240	270	300	360
y = csc θ								
y = sec θ								

3 SKETCH THE GRAPHS OF y = CSC θ AND y = SEC θ ON A SEPARATE COORDINATE SYSTEM.

4 CONSTRUCT A TABLE OF VALUES FOR y = COT θ AND SKETCH THE GRAPH.

Hint: USE THE TABLE OF VALUES FOR y = TAN θ

5 DISCUSS AND IDENTIFY THE VALUES OF θ WHERE CSC, SEC AND COT WILL BE UNDEFINED.

Exercise 5.11

1 SUPPOSE THE FOLLOWING POINTS LIE ON THE TERMINAL SIDE OF AN ANGLE θ. FIND THE COSECANT, SECANT AND COTANGENT FUNCTIONS OF θ:

A P (12, 5) **B** P (– 8, 15) **C** P (– 6, 8) **D** P (5, 3)

E P (2, 0) **F** P $\left(\frac{4}{5}, \frac{-3}{5}\right)$ **G** P $(\sqrt{2}, \sqrt{5})$ **H** P $(\sqrt{6}, \sqrt{3})$

2 COMPLETE EACH OF THE FOLLOWING:

A IF SIN θ IS – 0.35, THEN CSC θ IS ____. **B** IF SEC θ IS 2.6, THEN COS θ IS ____.

C IF CSC θ IS 30.5, THEN SIN θ IS ____. **D** IF TAN θ IS 1, THEN COT θ IS ____.

E IF TAN θ IS $\frac{\sqrt{3}}{3}$, THEN COT θ IS ____. **F** IF TAN θ IS 0, THEN COT θ IS ____.

3 FIND THE VALUES OF CSC θ, SEC θ AND COT θ, IF θ IN DEGREES IS:

A 30 **B** 45 **C** 60 **D** 120

E 150 **F** 210 **G** 240 **H** 300

I – 390 **J** – 405 **K** – 420 **L** 780.

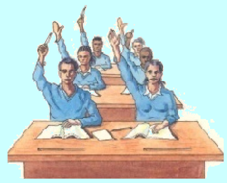
4 IF COT θ = $\frac{3}{8}$ AND θ IS IN THE FIRST QUADRANT, FIND THE OTHER FIVE TRIGONOMETRIC FUNCTIONS OF θ.

Co–functions

WHAT KINDS OF FUNCTIONS ARE CALLED CO-FUNCTIONS?

INORDER TO UNDERSTAND THE CONCEPT OF A CO-FUNCTION, TRY THE FOLLOWING ACTIVITY.

ACTIVITY 5.10



ABC IS A RIGHT ANGLE TRIANGLE. α AND β ARE ACUTE ANGLES. SINCE THEIR SUM IS 90°, THEY ARE complementary angles. FIND THE VALUES OF THE SIX TRIGONOMETRIC FUNCTIONS FOR BOTH α AND β AND COMPARE THE RESULTS.

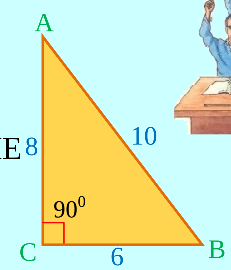


Figure 5.60

IDENTIFY THE FUNCTIONS THAT HAVE THE SAME VALUE.

FROM ACTIVITY 5.10, YOU MAY CONCLUDE THE FOLLOWING:

OBSERVE THAT $\triangle ABC$ IS A RIGHT ANGLE TRIANGLE WITH $m(\angle C) = 90^{\circ}$, $\alpha + \beta = 90^{\circ}$. THIS MEANS THE ACUTE ANGLES α AND β ARE complementary.

HENCE WE HAVE THE FOLLOWING RELATIONSHIP:

$\text{SIN} = \frac{a}{c} = \text{COS}$ $\qquad$ $\text{CSC} = \frac{c}{a} = \text{SEC}$

$\text{COS} = \frac{b}{c} = \text{SIN}$ $\qquad$ $\text{SEC} = \frac{c}{b} = \text{CSC}$

$\text{TAN} = \frac{a}{b} = \text{CO}$ $\qquad$ $\text{COT} = \frac{b}{a} = \text{TA}$

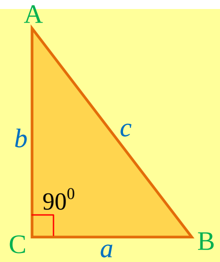


Figure 5.61

NOTE THAT, FOR THE TWO COMPLEMENTARY ANGLES α AND β.

✓ THE SINE OF ANY ANGLE IS EQUAL TO THE COSINE OF ITS COMPLEMENTARY ANGLE.

✓ THE TANGENT OF ANY ANGLE IS EQUAL TO THE COTANGENT OF ITS COMPLEMENTARY ANGLE.

✓ THE SECANT OF ANY ANGLE IS EQUAL TO THE COSECANT OF ITS COMPLEMENTARY ANGLE.

THUS, THE PAIR OF FUNCTIONS sine AND cosine ARE CALLED co-functions.

SIMILARLY, tangent AND cotangent, secant AND cosecant ARE ALSO CO-FUNCTIONS.

ANY TRIGONOMETRIC FUNCTION OF AN ACUTE ANGLE IS EQUAL TO THE CO-FUNCTION OF ITS COMPLEMENTARY ANGLE. THAT IS, IF $0^\circ \le \theta \le 90^\circ$, THEN

$\text{SIN}\,\theta = \text{COS}\,(90^0 - \theta)$ $\qquad$ $\text{CSC}\,\theta = \text{SEC}\,(90^0 - \theta)$

$\text{COS}\,\theta = \text{SIN}\,(90^0 - \theta)$ $\qquad$ $\text{SEC}\,\theta = \text{CSC}\,(90^0 - \theta)$

$\text{TAN}\,\theta = \text{COT}\,(90^0 - \theta)$ $\qquad$ $\text{COT}\,\theta = \text{TAN}\,(90^0 - \theta)$

EXAMPLE 6

A $\text{SIN}\,30^0 = \text{COS}\,60^0$ $\qquad$ **B** $\text{SEC}\,40^0 = \text{CSC}\,50^0$ $\qquad$ **C** $\text{TAN}\,\frac{\pi}{3} = \text{COT}\,\frac{\pi}{6}$

Exercise 5.12

1 FIND THE SIZE OF ACUTE ANGLE θ IN DEGREES IF:

A $\text{SIN}\,20^0 = \text{COS}\,\theta$ $\qquad$ **B** $\text{SEC}\,\theta = \text{CSC}\,80^0$ $\qquad$ **C** $\text{TAN}\,55^0 = \text{COT}\,\theta$

D $\text{COS}\,\frac{\pi}{9} = \text{SIN}\,\theta$ $\qquad$ **E** $\text{SEC}\,\theta = \text{CSC}\,\frac{5\pi}{12}$ $\qquad$ **F** $\text{COT}\,1^0 = \text{TAN}\,\theta$

2 ANSWER EACH OF THE FOLLOWING:

A IF $\text{COS}\,35^0 = 0.8387$, THEN $\text{SIN}\,55^0 =$ ______

B IF $\text{SIN}\,77^0 = 0.9744$, THEN $\text{COS}\,13^0 =$ ______

C IF $\text{TAN}\,45^0 = 1$, THEN $\text{COT}\,45^0 =$ ______

D IF $\text{SEC}\,15^0 = x$, THEN $\text{CSC}\,75^0 =$ ______

E IF $\text{CSC}\,\alpha = \frac{a}{b}$ AND $\text{SEC}\,\beta = \frac{a}{b}$, THEN $\alpha + \beta =$ ________

F IF $\text{COT}\,55^0 = y$ AND $\text{TAN}\,\theta = y$, THEN $\theta =$ __________

5.3 SIMPLE TRIGONOMETRIC IDENTITIES

Pythagorean identities

USING THE DEFINITIONS OF THE SIX TRIGONOMETRIC FUNCTIONS DISCUSSED SO FAR, IT IS POSSIBLE TO FIND SPECIAL RELATIONSHIPS THAT EXIST BETWEEN THEM.

LET BE AN ANGLE IN STANDARD POSITION AND P(x, y) BE A POINT ON THE TERMINAL SIDE OF (See FIGURE 5.62)

FROM PYTHAGORAS' THEOREM, WE KNOW THAT

$$x^2 + y^2 = r^2$$

IF WE DIVIDE BOTH SIDES BY r^2, WE HAVE

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$\therefore (\text{COS })^2 + (\text{SIN })^2 = 1$$

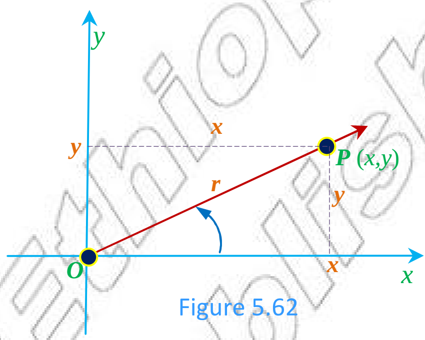


Figure 5.62

IF WE DIVIDE BOTH SIDES OF $x^2 + y^2 = r^2$ BY x^2, THEN WE HAVE

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2}$$

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

$$1 + (\text{TAN})^2 = (\text{SEC })^2$$

IF WE DIVIDE BOTH SIDES OF $x^2 + y^2 = r^2$ BY y^2, THEN WE HAVE

$$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{r^2}{y^2}$$

$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2$$

$$(\text{COT})^2 + 1 = (\text{CSC })^2$$

HENCE WE HAVE THE FOLLOWING RELATIONS:

$$\text{SIN}^2 + \text{COS}^2 = 1$$

$$1 + \text{TAN}^2 = \text{SEC}^2$$

$$1 + \text{COT}^2 = \text{CSC}^2$$

THE ABOVE RELATIONS ARE KNOWN AS PYTHAGOREAN IDENTITIES.

Note:

$(\text{SIN })^2 = \text{SIN}^2$ AND $(\text{COS })^2 = \text{COS}^2$, ETC.

EXAMPLE 1 IF SIN $= \frac{1}{2}$ AND IS IN THE FIRST QUADRANT, FIND THE VALUES OF THE OTHER FIVE TRIGONOMETRIC FUNCTIONS OF .

SOLUTION: FROM $\sin^2 + \cos^2 = 1$, WE HAVE

$\cos^2 = 1 - \sin^2$

$$\text{SO, } \cos = \sqrt{1-\sin^2} = \sqrt{1-\left(\frac{1}{2}\right)^2} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$\sec = \frac{1}{\cos} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}};\quad \csc = \frac{1}{\sin} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

FROM $1 + \tan^2 = \sec^2$, WE HAVE, $\tan^2 = \sec^2 - 1$

$$\text{SO } \tan = \sqrt{\sec^2 - 1} = \sqrt{\left(\frac{2}{\sqrt{3}}\right)^2 - 1} = \sqrt{\frac{4}{3} - 1} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

FROM $\cot^2 + 1 = \csc^2$, WE HAVE $\cot^2 = \csc^2 - 1$, THIS IMPLIES THAT

$$\cot = \sqrt{\csc^2 - 1} = \sqrt{2^2 - 1} = \sqrt{4 - 1} = \sqrt{3}.$$

Exercise 5.13

1 USING THE PYTHAGOREAN IDENTITIES FIND THE VALUES OF THE OTHER FIVE TRIGONOMETRIC FUNCTIONS IF:

A SIN $= \frac{15}{17}$ AND IS IN QUADRANT I.

B COS $= \frac{-4}{5}$ AND IS IN QUADRANT II

C COT $= \frac{7}{24}$ AND IS IN QUADRANT III.

D COS $= \frac{24}{25}$ AND IS IN QUADRANT IV.

2 REFERRING TO THE RIGHT ANGLE TRIANGLE ABC (*See* FIGURE 5.63), FIND:

A SIN **B** COS **C** SIN (90^0 –)

D COS (90^0 –) **E** CSC (90^0 –) **F** COT (90^0 –)

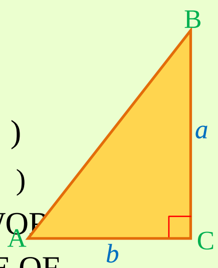


Figure 5.63

3 FILL IN THE BLANK SPACE WITH THE APPROPRIATE WORD.

A THE SINE OF AN ANGLE IS EQUAL TO THE COSINE OF ___

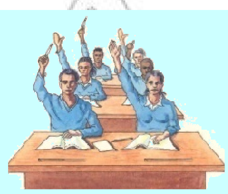
B THE COSECANT OF AN ANGLE IS EQUAL TO THE SECANT OF ___

C THE TANGENT OF AN ANGLE IS EQUAL TO THE ___ OF THE COMPLEMENTARY ANGLE.

Quotient identities

THE FOLLOWING ARE ADDITIONAL RELATIONSHIPS THAT CAN BE DERIVED FROM THE SIX TRIGONOMETRIC FUNCTIONS:

ACTIVITY 5.11



LET BE AN ANGLE IN STANDARD POSITION AND $P(x, y)$ BE A POINT ON THE TERMINAL SIDE OF (*See* FIGURE 5.64).

THEN ANSWER THE FOLLOWING:

A WHAT ARE THE VALUES OF SIN, COS, TAN AND COT?

B HOW DO THE VALUES $\frac{\text{SIN}}{\text{CO}}$ AND TAN COMPARE?

C HOW DO THE VALUES $\frac{\text{COS}}{\text{SIN}}$ AND COT COMPARE?

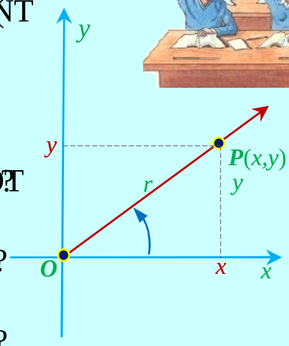


Figure 5.64

REFERRING TO FIGURE 5.64, WE CAN DERIVE THE FOLLOWING RELATIONSHIPS BETWEEN THE SIX TRIGONOMETRIC FUNCTIONS:

$$\text{SIN} = \frac{y}{r} \text{ AND COS} = \frac{x}{r}. \text{ FROM THIS WE HAVE, } \frac{\text{SIN}}{\text{CO}} = \frac{\left(\frac{y}{r}\right)}{\left(\frac{x}{r}\right)} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x} = \text{TAN}$$

$$\text{SIMILARLY, } \frac{\text{COS}}{\text{SIN}} = \frac{\left(\frac{x}{r}\right)}{\left(\frac{y}{r}\right)} = \frac{x}{r} \times \frac{r}{y} = \frac{x}{y} = \text{COT}$$

HENCE THE RELATIONS:

$\text{TAN} = \frac{\text{SIN}}{\text{COS}}$ AND $\text{COT} = \frac{\text{COS}}{\text{SIN}}$ WHICH ARE KNOWN AS **quotient** IDENTITIES.

EXAMPLE 2 IF $\text{SIN} = \frac{4}{5}$ AND $\text{COS} = \frac{3}{5}$, THEN FIND TAN AND COT

SOLUTION: FROM QUOTIENT IDENTITY TAN $= \frac{\text{SIN}}{\text{CO}} = \frac{\left(\frac{4}{5}\right)}{\left(\frac{3}{5}\right)} = \frac{4}{3}$

$$\text{COT} = \frac{\text{COS}}{\text{SIN}} = \frac{\left(\frac{3}{5}\right)}{\left(\frac{4}{5}\right)} = \frac{3}{4}$$

Note: AN IDENTITY IS AN EQUATION THAT IS TRUE FOR ALL VALUES OF THE VARIABLE FOR WHICH BOTH SIDES OF THE EQUATION ARE DEFINED.

ALL IDENTITIES ARE EQUATIONS BUT ALL EQUATIONS ARE NOT IDENTITIES. THIS IS BECAUSE, UNLIKE IDENTITIES , EQUATIONS MAY NOT BE TRUE FOR SOME VALUES IN TH

FOR EXAMPLE CONSIDER THE EQUATION SIN = COS

FOR MOST VALUES OF , THIS EQUATION IS NOT TRUE (*for instance*, SIN $30^\circ \neq$ COS 30°)

HENCE THE EXPRESSION SIN = COS REPRESENTS A SIMPLE TRIGONOMETRIC EQUATION, BUT NOT AN IDENTITY.

Group Work 5.6

USE THE PYTHAGOREAN AND QUOTIENT IDENTITIES TO SOLVE EACH OF THE FOLLOWING:

1. COS $\alpha = \frac{-4}{5}$ AND α IS IN QUADRANT II. FIND TAN α AND COT α.
2. SIN $\alpha = \frac{8}{17}$ AND α IS IN QUADRANT I. FIND TAN α AND COT α.
3. SIN $330^\circ = -\frac{1}{2}$. FIND TAN 330° AND COT 330°.
4. COS $150^\circ = -\frac{\sqrt{3}}{2}$. FIND TAN 150° AND COT 150°.
5. SEC $60^\circ = 2$. FIND TAN 60° AND COT 60°.
6. SUPPOSE α IS AN ACUTE ANGLE SUCH THAT SIN $\alpha = x$ AND SIN $(90^\circ - \alpha) = y$; FIND TAN $(90^\circ - \alpha)$ AND COT $(90^\circ - \alpha)$.

Using tables of the trigonometric functions

SO FAR YOU HAVE SEEN HOW TO DETERMINE THE VALUES OF TRIGONOMETRIC FUNCTIONS OF SOME SPECIAL ANGLES. THE SAME METHODS CAN IN THEORY BE APPLIED TO ANY ANGLE RESULTS FOUND IN THIS WAY ARE APPROXIMATIONS. THEREFORE WE USE PUBLISH VALUES, WHERE VALUES ARE GIVEN TO FOUR DECIMAL PLACES OF ACCURACY.

SINCE THE TRIGONOMETRIC FUNCTIONS OF A POSITIVE ACUTE ANGLE AND THE CORRESPONDING CO-FUNCTIONS OF THE COMPLEMENTARY ANGLE $(90^\circ -)$ ARE EQUAL, TRIGONOMETRIC TABLES ARE OFT CONSTRUCTED ONLY FOR VALUES OF BETWEEN 0° AND 45°

TO FIND THE TRIGONOMETRIC FUNCTIONS OF ANGLES BETWEEN 45° AND 90°, A TABLE CONSTRUCTED FOR VALUES OF BETWEEN 0° AND 45° IS USED BY READING FROM BOTTOM UP. CORRESPONDING TO EACH ANGLE BETWEEN 0° AND 45° LISTED IN THE LEFT HAND COLUMN, THE COMPLEMENTARY ANGLE $(90^\circ -)$ IS LISTED IN THE RIGHT HAND COLUMN. CORRESPONDING TO EACH TRIGONOMETRIC FUNCTION LISTED AT THE TOP, THE CO-FUNCTION IS LISTED AT THE BOTTOM. THUS, FOR ANGLES F THE TRIGONOMETRIC FUNCTIONS ARE READ USING THE BOTTOM ROW AND THE RIGHT HA

(*A part of the trigonometric table is given below for your reference*).

	sin	cos	tan	cot	
0°	0.0000	1.0000	0.0000	–	90°
1°	0.0175	0.9998	0.0175	57.29	89°
2°	--------	--------	--------	--------	88°
. . .					. . .
5°	0.0872	--------	0.0875	--------	85°
. . .					. . .
45°	--------	--------	--------	--------	45°
	cos	sin	cot	tan	

FOR INSTANCE, SIN 5° AND COS 85° ARE BOTH FOUND AT THE SAME PLACE IN THE TABLE AND EA APPROXIMATELY EQUAL TO 0.0872. SIMILARLY, TAN 5° = COT 85° = 0.0875, ETC.

EXAMPLE 3 USE THE TABLE GIVEN AT THE END OF THE BOOK TO FIND THE APPROXIMATE VALUES OF:

A COS 20° **B** COT 50°

SOLUTION:

A SINCE 20° < 45°, WE BEGIN BY LOCATING 20° IN THE VERTICAL COLUMN ON THE LEFT SIDE OF THE DEGREE TABLE. THEN WE READ THE ENTRY 0.9397 UNDER THE C LABELLED COS AT THE TOP.

∴ COS 20° = 0.9397 .

B SINCE 50° > 45°, WE USE THE VERTICAL COLUMN ON THE RIGHT SIDE (REA UPWARD) TO LOCATE 50° AND READ ABOVE THE BOTTOM CAPTION "COT" TO G 0.8391;

∴ COT 50° = 0.8391.

EXAMPLE 4 FIND SO THAT:

A SEC = 1.624 **B** SIN = 0.5831

SOLUTION: FINDING AN ANGLE WHEN THE VALUE OF ONE OF ITS FUNCTIONS IS GIVEN IS THE REVERSE PROCESS OF THAT ILLUSTRATED IN THE ABOVE EXAMPLE.

A GIVEN SEC = 1.624, LOOKING UNDER THE SECANT COLUMN OR ABOVE THE SE COLUMN, WE FIND THE ENTRY 1.624 ABOVE THE SECANT COLUMN AN CORRESPONDING VALUE OF IS 52°. THEREFORE, = 52°.

B REFERRING TO THE "SINE" COLUMNS OF THE TABLE, 0.5831 DOES NOT APPEAR THERE. THE TWO VALUES IN THE TABLE CLOSEST TO 0.5831 (ONE SMA ONE LARGER) ARE 0.5736 AND 0.5878. THESE VALUES CORRESPOND TO 35° AND 36°, RESPECTIVELY. AS SHOWN BELOW, THE DIFFERENCE BETWEEN AND THE VALUE OF SIN 36° IS SMALLER THAN THE DIFFERENCE BETWEEN SIN 35° AND SIN. THEREFORE USE THE VALUE 36° FOR BECAUSE SIN IS CLOSER TO SIN 36° THAN IT IS TO SIN 35°.

SIN = 0.5831	SIN 36° = 0.5878
SIN 35° = 0.5736	SIN = 0.5831
DIFFERENCE = 0.0095	DIFFERENCE = 0.0047

∴ = 36° (NEAREST DEGREE).

THE FOLLOWING EXAMPLES ILLUSTRATE HOW TO DETERMINE THE VALUES OF TI FUNCTIONS FOR ANGLES MEASURED IN DEGREES (OR RADIANS) WHOSE MEASURES AR

0° AND 90° (OR 0 AND $\frac{}{2}$).

EXAMPLE 5 USE THE NUMERICAL TABLE, REFERENCE ANGLES, TRIGONOMETR FUNCTIONS OF NEGATIVE ANGLES AND PERIODICITY OF THE FUNCTIONS TO DETERMINE T EACH OF THE FOLLOWING:

A SIN 236° **B** COS 693°

SOLUTION:

A TO FIND SIN 236°, FIRST WE CONSIDER THE QUADRANT THAT THE ANGLE 236° BELONGS TO. THIS IS DONE BY PLACING THE ANGLE IN STANDARD POSITION AS SHO FIGURE 5.65. WE SEE THAT THE ANGLE 236° LIES IN QUADRANT III SO THAT THE SINE VALUE IS NEGATIVE. THE REFERENCE ANGLE CORRESPONDING TO 236° IS

$_R$ = 236° – 180° = 56°. THUS, SIN 236° = – SIN 56°.

SINCE 56° > 45°, WE LOCATE 56° IN THE VERTICAL COLUMN ON THE RIGHT SIDE OF THE TRIGONOMETRIC TABLE. LOOKING IN THE VERTICAL COLUMN ABOVE THE BOTTOM CAPTION "SIN", WE SEE THAT SIN 56° = 0.8290.

SO SIN 236° = – SIN 56° = – 0.8290.

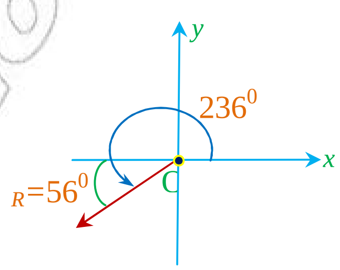


Figure 5.65

b TO FIND THE VALUE OF COS 693°, FIRST OBSERVE THAT 693° IS GREATER THAN 360°. THE PERIOD OF COSINE FUNCTION IS 360°. DIVIDING 693° BY 360° WE OBTAIN

$$693° = 1 \times 360° + 333°$$

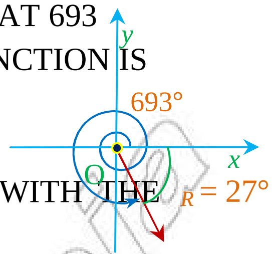


Figure 5.66

THIS MEANS THAT THE 693° ANGLE IS CO TERMINAL WITH THE 333° ANGLE. I.E., COS 693° = COS 333°.

SINCE THE TERMINAL SIDE OF 333° IS IN QUADRANT IV, THE REFERENCE ANGLE IS θ_R = 360° – 333° = 27° (*See* Figure 5.66)

COSINE IS POSITIVE IN QUADRANT IV, SO COS 333° = COS 27° = 0.8910 .

HENCE, COS 693° = 0.8910.

Exercise 5.14

1 USING TRIGONOMETRIC TABLE, FIND:

a SIN 59° **b** COS 53° **c** TAN 36° **d** SEC 162°

e SIN 593° **f** TAN 593° **g** COS (–143°)

2 IN EACH OF THE FOLLOWING PROBLEMS, FIND THE ANGLE A CORRECT TO THE NEAREST DEGREE:

a SIN A = 0.5299 **b** COS A = 0.6947 **c** TAN A = 1.540

d CSC A = 1.000 **e** SEC A = 2.000 **f** COT A = 1.808

5.4 REAL LIFE APPLICATION PROBLEMS

EVEN THOUGH TRIGONOMETRY WAS ORIGINALLY USED TO RELATE THE ANGLES OF A LENGTHS OF THE SIDES OF A TRIANGLE, TRIGONOMETRIC FUNCTIONS ARE IMPORTANT STUDY OF TRIANGLES BUT ALSO IN MODELING MANY PERIODIC PHENOMENA IN REA SECTION YOU WILL SEE SOME OF THE REAL LIFE APPLICATIONS OF TRIGONOMETRY.

Solving right-angled triangles

MANY APPLICATIONS OF TRIGONOMETRY INVOLVE "SOLVING A TRIANGLE". A TRIANGLE HAS BASICALLY SEVEN COMPONENTS; NAMELY THREE SIDES, THREE ANGLES AND AN AREA. THUS, SOLV MEANS TO FIND THE LENGTHS OF THE THREE SIDES, THE MEASURES OF ALL THE THREE MEASURE OF ITS AREA.

Revision of the properties of right angle triangles

WE ALREADY KNOW THAT, FOR A GIVEN RIGHT ANGLED TRIANGLE, THE hypotenuse (HYP) IS THE SIDE WHICH IS OPPOSITE THE RIGHT ANGLE AND IS THE LONGEST SIDE OF THE TRIANGLE. FOR THE ANGLE MARKED IN FIGURE 5.67:

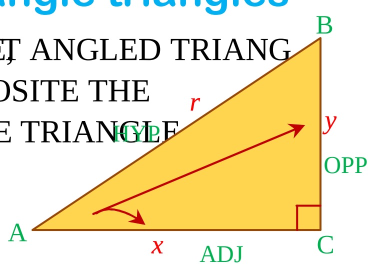


Figure 5.67

- ✓ $\overline{BC}$ IS THE SIDE opposite (OPP) ANGLE
- ✓ $\overline{AC}$ IS THE SIDE adjacent (ADJ) ANGLE.

HENCE,

1	$x^2 + y^2 = r^2$	**2**	$\sin = \frac{y}{r}$ $\csc = \frac{r}{y} = \frac{1}{\sin}$ $\cos = \frac{x}{r}$ $\sec = \frac{r}{x} = \frac{1}{\cos}$ $\tan = \frac{y}{x}$ $\cot = \frac{x}{y} = \frac{1}{\tan}$
3	$\sin^2 + \cos^2 = 1$ $1 + \tan^2 = \sec^2$ $1 + \cot^2 = \csc^2$	**4**	$\tan = \frac{\sin}{\cos}$ $\cot = \frac{\cos}{\sin}$

EXAMPLE 1 SOLVE THE RIGHT-ANGLED TRIANGLE WITH AN ACUTE ANGLE OF 25° AND HYPOTENUSE OF LENGTH 10 CM.

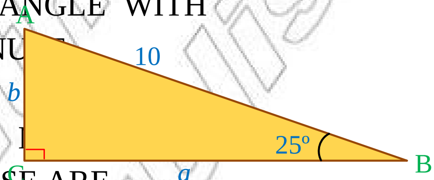


Figure 5.68

SOLUTION: IT IS REQUIRED TO FIND THE UNKNOWN ELEMENTS OF THE TRIANGLE. THESE ARE

A $m(\angle A)$ **B** LENGTH OF SIDE BC

C LENGTH OF SIDE AC **D** THE AREA OF THE TRIANGLE

DRAW THE TRIANGLE AND LABEL ALL KNOWN PARTS (*See* Figure 5.68)

A $m(\angle A) = 90^\circ - 25^\circ = 65^\circ$

B TO FIND a, OBSERVE THAT THE SIDE $\overline{BC}$ IS OPPOSITE TO THE 65° ANGLE, AND THE LENGTH OF THE HYPOTENUSE IS 10 CM. SO $\sin 65^\circ = \frac{a}{10}$

MULTIPLYING BOTH SIDES OF THE EQUATION BY 10, WE OBTAIN

$$a = 10 \times \sin 65^\circ$$

USING THE TRIGONOMETRIC TABLE, WE HAVE

$$a = 10 \times \sin 65^\circ \approx 10 \times 0.9063 = 9.063 \text{ CM}$$

C TO FIND b, WE CAN USE THE PYTHAGOREAN THEOREM OR THE SINE FUNCTION.

$$\sin 25^\circ = \frac{b}{10}$$

MULTIPLYING BOTH SIDES BY 10 WE OBTAIN $b = 10 \times \sin 25^\circ$

USING TRIGONOMETRIC TABLE WE HAVE $b = 10 \times \sin 25^\circ \approx 10 \times (0.4226) \approx 4.226$ CM.

D AREA OF $\triangle ABC = \frac{1}{2}ab \approx \frac{1}{2} \times 9.063 \times 4.226 \approx 19.150 \text{ CM}^2$

EXAMPLE 2 SOLVE THE RIGHT ANGLE TRIANGLE WHOSE HYPOTENUSE IS 20 UNITS AND ONE OF THE LEGS IS 17 UNITS.

SOLUTION: THE MISSING ELEMENTS OF THE TRIANGLE ARE

A $m(\angle A)$ **C** LENGTH OF SIDE AC

B $m(\angle B)$ **D** THE AREA OF THE TRIANGLE

DRAW THE TRIANGLE (See FIGURE 5.69).

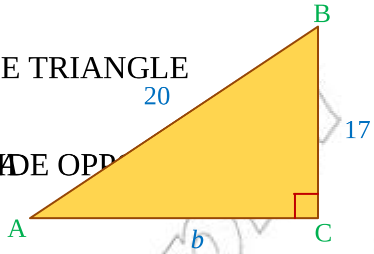


Figure 5.69

A SINCE THE HYPOTENUSE AND THE SIDE OPP... ARE GIVEN,

$$\sin A = \frac{17}{20} = 0.8500$$

THUS, FROM THE TRIGONOMETRIC TABLE, WE SEE THAT $m(\angle A) \approx 58^{O}$

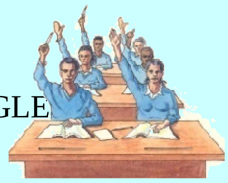
B $m(\angle B) = 90^{O} - m(\angle A) = 90^{O} - 58^{O} = 32^{O}$

C TO FIND b, USE $\cos A = \dfrac{b}{20}$ WHICH GIVES

$$b = 20 \cos A \approx 20 \cos 58^{O} \approx 20(0.5299) \approx 10.598$$

D AREA OF $\triangle ABC = \frac{1}{2} \times b \times 17 = \frac{1}{2} \times 10.598 \times 17 = 90.083$ UNITS^2

ACTIVITY 5.12



1. SOLVE THE RIGHT ANGLED TRIANGLE ABC WITH THE RIGHT ANGLE AT B, $AB = 2$ CM AND $BC = 3$ CM.
2. SOLVE THE RIGHT ANGLED TRIANGLE ABC WITH THE RIGHT ANGLE AT B, $m(\angle A) = 24°$ AND $AB = 20$ CM.

Angle of elevation and angle of depression

THE **line of sight** OF AN OBJECT IS THE LINE JOINING THE EYE OF AN OBSERVER AND THE OBJECT. IF THE OBJECT IS ABOVE THE HORIZONTAL PLANE THROUGH THE EYE OF THE OBSER BETWEEN THE LINE OF SIGHT AND THIS HORIZONTAL PLANE IS CALLED THE **angle of elevation** (*See* FIGURE 5.70). IF THE OBJECT IS **below** THIS HORIZONTAL PLANE, THE ANGLE IS THEN CALL THE **angle of depression**.

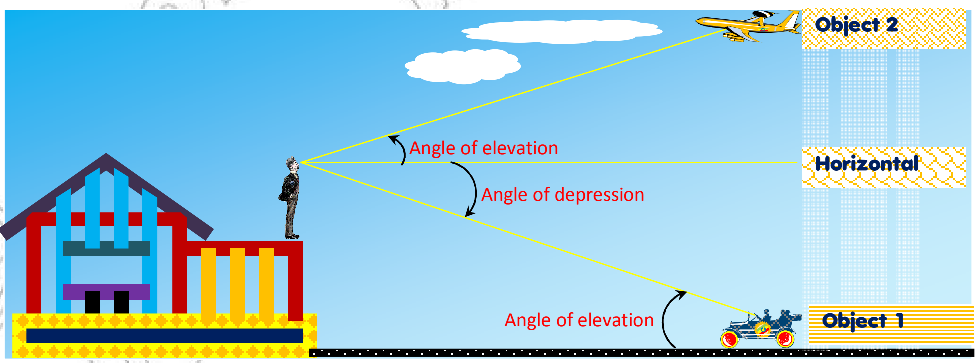


Figure 5.70

EXAMPLE 3 Find the height of a tree which casts a shadow of 12.4 m when the angle of elevation of the sun is 52°.

SOLUTION: Let h be the height of the tree in metres. For the 52° angle, the opposite side is h and the adjacent side 12.4 m.

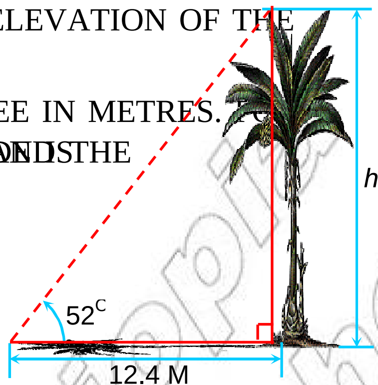


Figure 5.71

Therefore, $\tan 52^{\circ} = \frac{h}{12.4}$

$\therefore h = 12.4 \times \tan 52^{\circ} \approx 15.9$ m.

Therefore, the tree is 15.9 m high.

EXAMPLE 4 From the top of a building, the angle of depression of a point on the ground 7 m away from the base of the building is 60°. Find the height of the building.

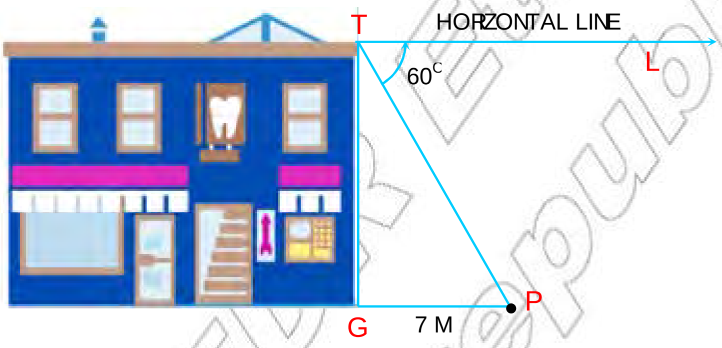


Figure 5.72

SOLUTION: In Figure 5.72, T is a point on the top of the building, P is the point on the ground, and $\overrightarrow{TL}$ is a horizontal ray through T in the plane of GTP.

$m(\angle GPT) = m(\angle LTP) = 60^{\circ}$ (why?)

$$\frac{GT}{PG} = \tan(\angle GPT) \Rightarrow \tan 60^{\circ} = \frac{GT}{7} \Rightarrow GT = 7\tan 60^{\circ} \approx 7 \times 1.732 \approx 12 \text{ m.}$$

Therefore, the height of the building is about 12 metres.

EXAMPLE 5 A person standing on the edge of one bank of a canal observes a lamp post on the edge of the other bank of the canal. The person's eye is 152 cm above the ground. The angle of elevation from eye level to the top of the lamp post is 12°, and the angle of depression from eye level to the bottom of the lamp post is 7°. How high is the lamp post? How wide is the canal? (See Figure 5.73A.)

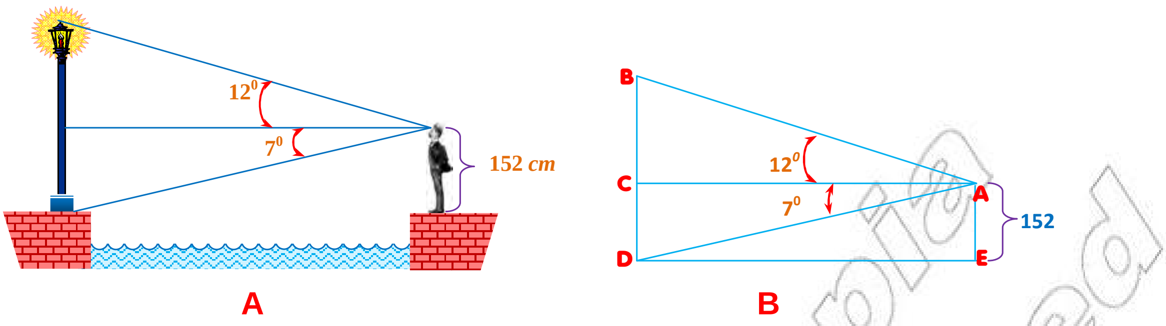


Figure 5.73

SOLUTION: CONSIDERING THE ESSENTIAL INFORMATION, WE CAN DRAW THE DIAGRAM AS IN FIGURE 5.73B.

WE WANT TO FIND THE HEIGHT OF THE LAMP POST BD AND THE WIDTH OF THE CANAL AC. THE EYE LEVEL HEIGHT AE OF THE OBSERVER IS 152CM. SINCE $\overline{AC}$ AND $\overline{ED}$ ARE PARALLEL, $\overline{CD}$ ALSO HAS LENGTH 152 CM. IN THE RIGHT ANGLE TRIANGLE ADC, WE KNOW THAT THE SIDE CD IS OPPOSITE TO THE ANGLE OF 7°.

SO, $\tan 7° = \dfrac{opp}{adj} = \dfrac{152}{AC}$ GIVING $AC = \dfrac{152}{\tan 7°}$

THEREFORE, $AC = \dfrac{152}{\tan 7°} = \dfrac{152}{0.1228} = 1237.79$ CM

SO THE CANAL IS APPROXIMATELY 12.4 METRES WIDE.

NOW, USING THE RIGHT TRIANGLE ACB, WE SEE THAT

$$\tan 12° = \frac{opp}{adj} = \frac{BC}{AC} = \frac{BC}{1237.79}$$

THEREFORE, $BC = 1237.79 \times \tan 12° = 1237.79 \times 0.2126 = 263.15$ CM.

SO THE HEIGHT OF THE LAMP POST BD IS

$BC + CD = 263.15 + 152 = 415.15$ CM ≈ 4.15 M

Exercise 5.15

1 IN PROBLEMS A TO F, ΔABC IS A RIGHT ANGLE TRIANGLE WITH $m(\angle C) = 90°$. LET a, b, c BE ITS SIDES WITH c THE LENGTH OF ITS HYPOTENUSE, a ITS SIDE LENGTH OPPOSITE ANGLE A AND b ITS SIDE LENGTH OPPOSITE B. USING THE INFORMATION BELOW, FIND THE MISSING ELEMENTS OF EACH RIGHT ANGLE TRIANGLE , GIVING ANSWERS CORRECT TO THE NUMBER.

A	$m(\angle B) = 50°$ AND $c = 20$ UNITS	**B**	$m(\angle A) = 54°$ AND $a = 12$ UNITS
C	$m(\angle A) = 36°$ AND $b = 8$ UNITS	**D**	$m(\angle B) = 55°$ AND $a = 10$ UNITS
E	$m(\angle A) = 38°$ AND $c = 20$ UNITS	**F**	$m(\angle A) = 17°$ AND $a = 14$ UNITS.

2 **A** A LADDER 6 METRES LONG LEANS AGAINST A BUILDING. IF THE ANGLE FORMED BY THE LADDER AND THE GROUND IS 66°, HOW FAR FROM THE BUILDING IS THE FOOT OF THE LADDER?

B A MONUMENT IS 50 METRES HIGH. WHAT IS THE LENGTH OF THE SHADOW CAST BY THE MONUMENT IF THE ANGLE OF ELEVATION OF THE SUN IS 60°?

C WHEN THE SUN IS 35° ABOVE THE HORIZON, HOW LONG IS THE SHADOW CAST BY A BUILDING 15 METRES HIGH?

D FROM AN OBSERVER O, THE ANGLES OF ELEVATION OF THE BOTTOM AND THE TOP OF A FLAGPOLE ARE 40° AND 45° RESPECTIVELY. FIND THE HEIGHT OF THE FLAGPOLE.

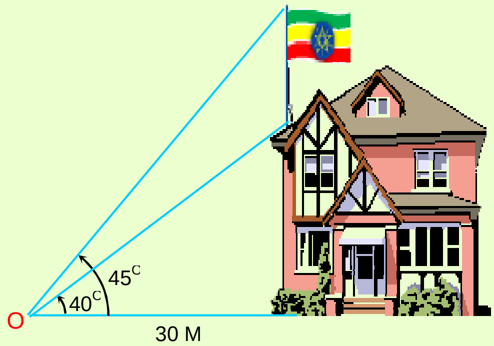


Figure 5.74

E FROM THE TOP OF A CLIFF 200 METRES ABOVE SEA LEVEL, THE ANGLES OF DEPRESSION TO TWO FISHING BOATS ARE 40° AND 45° RESPECTIVELY. HOW FAR APART ARE THE BOATS?

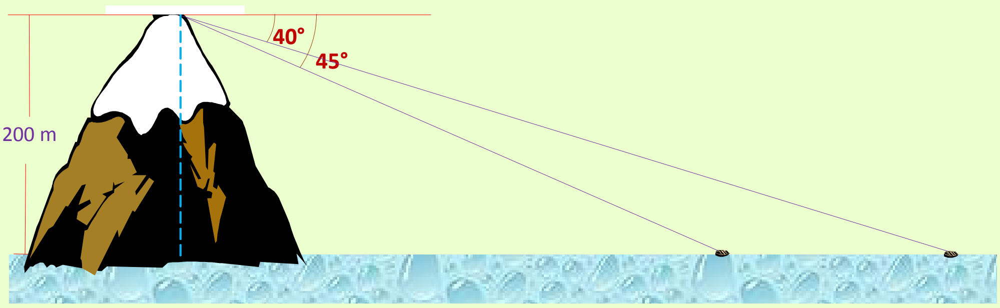


Figure 5.75

F A SURVEYOR STANDING AT A NOTICES TWO OBJECTS B AND C ON THE OPPOSITE SIDE OF A CANAL. THE OBJECTS ARE 120 M APART. IF THE ANGLE OF SIGHT BETWEEN THEM IS 37°, HOW WIDE IS THE CANAL?

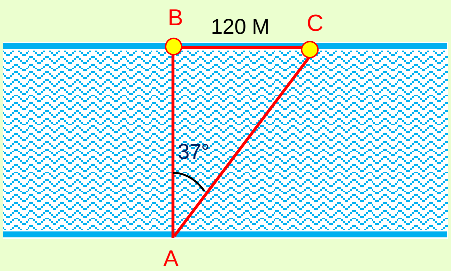


Figure 5.76

Key Terms

angle in standard position	negative angle	radian
angle of depression	period	reference angle
angle of elevation	periodic function	special angle
co-function	positive angle	supplementary angles
complementary angles	pythagorean identity	trigonometric function
co-terminal angles	quadrantal angle	trigonometry
degree	quotient identity	unit circle

Summary

1 AN ANGLE IS DETERMINED BY THE ROTATION OF A RAY ABOUT ITS VERTEX FROM AN INITIAL POSITION TO A TERMINAL POSITION.

2 AN ANGLE IS positive FOR ANTICLOCKWISE ROTATION AND negative FOR CLOCKWISE ROTATION.

3 AN ANGLE IN THE COORDINATE PLANE IS IN standard Position, IF ITS VERTEX IS AT THE ORIGIN AND ITS INITIAL SIDE ALONG THE POSITIVE x-AXIS.

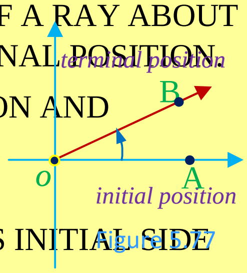


Figure 5.77

4 RADIAN MEASURE OF ANGLES:

2 RADIANS = 360^0 RADIANS = 180^0

5 TO CONVERT DEGREES TO RADIANS, MULTIPLY BY $\frac{\quad}{180^{\circ}}$.

6 TO CONVERT RADIANS TO DEGREES, MULTIPLY BY $\frac{180^{\circ}}{\quad}$.

7 IF IS AN ANGLE IN STANDARD POSITION AND $P(x, y)$ IS A POINT ON THE TERMINAL SIDE OF , OTHER THAN THE ORIGIN $O(0, 0)$, AND r IS THE DISTANCE OF P FROM THE ORIGIN O, THEN

SIN $= \frac{y}{r}$ CSC $= \frac{r}{y} = \frac{1}{\text{SI}}$

COS $= \frac{x}{r}$ SEC $= \frac{r}{x} = \frac{1}{\text{CO}}$

TAN $= \frac{y}{x}$ COT $= \frac{x}{y} = \frac{1}{\text{TA}}$

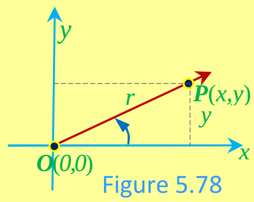


Figure 5.78

$r = \sqrt{x^2 + y^2}$ (PYTHAGORAS RULE)

8 Signs of sine, cosine and tangent functions:

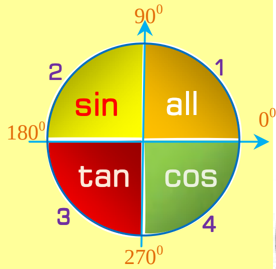


Figure 5.79

- ✓ IN THE FIRST QUADRANT all the three TRIGONOMETRIC FUNCTIONS ARE POSITIVE.
- ✓ IN THE SECOND QUADRANT sine IS ONLY POSITIVE.
- ✓ IN THE THIRD QUADRANT tangent IS ONLY POSITIVE.
- ✓ IN THE FOURTH QUADRANT cosine IS ONLY POSITIVE.

All Students Take Chemistry

9 Functions of negative angles:

IF IS AN ANGLE IN STANDARD POSITION, THEN

SIN(–) = – SIN   COS(–) = COS   TAN(–) = – TAN

10 Complementary angles:

TWO ANGLES ARE SAID TO BE COMPLEMENTARY, IF THEIR SUM IS EQUAL TO 90°

IF α AND β ARE ANY TWO COMPLEMENTARY ANGLES, THEN

SIN α = COS β COS α = SIN β TAN α = $\frac{1}{\text{TAN}}$

11 Reference angle R:

IF IS AN ANGLE IN STANDARD POSITION WHOSE TERMINAL SIDE DOES NOT LIE ON COORDINATE AXIS, THEN THE reference angle R FOR IS THE positive acute angle FORMED BY THE TERMINAL SIDE AND THE AXIS.

12 THE VALUES OF THE TRIGONOMETRIC FUNCTION OF A GIVEN ANGLE AND THE VALUES OF THE CORRESPONDING TRIGONOMETRIC FUNCTIONS OF THE REFERENCE ANGLE R ARE THE SAME IN ABSOLUTE VALUE BUT THEY MAY DIFFER IN SIGN

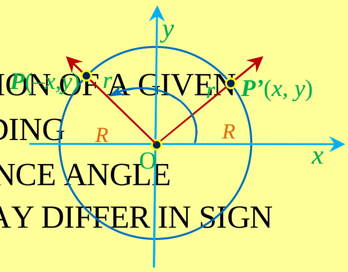


Figure 5.80

13 Supplementary angles:

TWO ANGLES ARE SAID TO BE SUPPLEMENTARY, IF THEIR SUM IS EQUAL TO 180°. IF IS A SECOND QUADRANT ANGLE, THEN ITS SUPPLEMENT WILL BE (180° –)

SIN = SIN (180° –),

COS = – COS (180° –) ,

TAN = – TAN (180° –)

14 CO-TERMINAL ANGLES ARE ANGLES IN STANDARD POSITION (ANGLES WITH THE INITIAL SIDE ON THE POSITIVE x-AXIS) THAT HAVE A COMMON TERMINAL SIDE.

15 CO-TERMINAL ANGLES HAVE THE SAME TRIGONOMETRIC VALUES

16 THE DOMAIN OF THE SINE FUNCTION IS THE SET OF ALL REAL NUMBERS.

17 THE RANGE OF THE SINE FUNCTION IS $\{y \mid -1 \le y \le 1\}$.

18 THE GRAPH OF THE SINE FUNCTION REPEATS ITSELF EVERY 360° OR 2 RADIANS.

19 THE DOMAIN OF THE COSINE FUNCTION IS THE SET OF ALL REAL NUMBERS.

20 THE RANGE OF THE COSINE FUNCTION IS $\{y \mid -1 \le y \le 1\}$.

21 THE GRAPH OF THE COSINE FUNCTION REPEATS ITSELF EVERY 360° OR 2 RADIANS.

22 THE DOMAIN OF THE TANGENT FUNCTION IS $\left\{x \mid x \neq n\frac{\ }{2}, \text{ WHERE } n \text{ IS AN ODD INTEGER}\right\}$

23 THE RANGE OF $y = \text{TAN}\ x$ IS THE SET OF ALL REAL NUMBERS.

24 THE TANGENT FUNCTION HAS PERIOD 180° OR RADIANS.

25 THE GRAPH OF $y = \text{TAN}\ x$ IS INCREASING FOR $-\frac{\ }{2} < x < \frac{\ }{2}$.

26 ANY TRIGONOMETRIC FUNCTION OF AN ACUTE ANGLE IS EQUAL TO THE CO-FUNCTION OF ITS COMPLEMENTARY ANGLE.

THAT IS, IF $0^{\circ} \le \ \le 90^{\circ}$, THEN

SIN = COS $(90^{\circ} - \ )$	CSC = SEC $(90^{\circ} - \ )$
COS = SIN $(90^{\circ} - \ )$	SEC = CSC $(90^{\circ} - \ )$
TAN = COT $(90^{\circ} - \ )$	COT = TAN $(90^{\circ} - \ )$

27 **Reciprocal relations:**

$$\text{CSC} = \frac{1}{\text{SIN}}, \quad \text{SEC} = \frac{1}{\text{COS}}, \quad \text{COT} = \frac{1}{\text{TAN}}$$

28 **Pythagorean identities:**

$$\text{SIN}^2 + \text{COS}^2 = 1 \qquad 1 + \text{TAN}^2 = \text{SEC}^2 \qquad \text{COT}^2 + 1 = \text{CSC}^2$$

29 **Quotient identities:**

$$\text{TAN} = \frac{\text{SIN}}{\text{COS}} \qquad \text{COT} = \frac{\text{COS}}{\text{SIN}}$$

Review Exercises on Unit 5

1 INDICATE TO WHICH QUADRANT EACH OF THE FOLLOWING ANGLES BELONG:

A 225° **B** 333° **C** -300° **D** 610°

E -700° **F** 900° **G** -765° **H** -1238°

I 1440° **J** 2010°.

2 FIND TWO CO-TERMINAL ANGLES (ONE POSITIVE AND ONE NEGATIVE) FOR EACH OF THE FOLLOWING ANGLES:

A 80° **B** 140° **C** 290° **D** 375° **E** 2900°

F -765° **G** -900° **H** -1238° **I** -1440° **J** -2010°.

3 CONVERT EACH OF THE FOLLOWING TO RADIANS:

A 40° **B** 75° **C** 240° **D** 330° **E** -95°

F -180° **G** -220° **H** -420° **I** -3060°.

4 CONVERT EACH OF THE FOLLOWING ANGLES IN RADIANS TO DEGREES:

A $\frac{2}{6}$ **B** $\frac{-2}{3}$ **C** $\frac{7}{18}$ **D** $\frac{43}{6}$

E $-\frac{4}{9}$ **F** 5 **G** $\frac{-3}{12}$ **H** $\frac{-}{24}$.

5 USE A UNIT CIRCLE TO FIND THE VALUES OF SINE, COSINE AND TANGENT WHEN IS:

A 810° **B** -450° **C** 900° **D** -630°

E 990° **F** -990° **G** 1080° **H** -1170°

6 FIND THE VALUES OF SINE, COSINE AND TANGENT WHEN IN RADIANS IS:

A $\frac{5}{6}$ **B** $\frac{7}{6}$ **C** $\frac{4}{3}$ **D** $\frac{3}{2}$

E $\frac{5}{3}$ **F** $\frac{-5}{3}$ **G** $\frac{-7}{4}$ **H** $\frac{-11}{6}$.

7 STATE WHETHER EACH OF THE FOLLOWING FUNCTIONAL VALUES ARE POSITIVE OR NEGATIVE:

A SIN 310° **B** COS 220° **C** COS (-220°) **D** TAN 765°

E SIN (-90°) **F** SEC (-70°) **G** TAN 327° **H** COT $\frac{5}{3}$

I CSC 1387° **J** SIN $\left(\frac{-11}{6}\right)$

8 GIVE A REFERENCE ANGLE FOR EACH OF THE FOLLOWING;

A 140° **B** 260° **C** 355° **D** 414°

E -190° **F** -336° **G** 1238° **H** -1080°.

9 REFERRING TO THE VALUES GIVEN IN THE TABLE BELOW FOR $0 \le \theta \le 360^\circ$ ROUGHLY SKETCH THE GRAPHS OF THE SINE, COSINE AND TANGENT FUNCTIONS.

Degrees	Radians	sin	cos	tan	cot	sec	csc
0º	0	0	1	0	UNDEFINED	1	UNDEFINED
30º	$\frac{}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45º	$\frac{}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60º	$\frac{}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
90º	$\frac{}{2}$	1	0	UNDEFINED	0	UNDEFINED	1
120º	$\frac{2}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{-\sqrt{3}}{3}$	–2	$\frac{2\sqrt{3}}{3}$
135º	$\frac{3}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	–1	–1	$-\sqrt{2}$	$\sqrt{2}$
150º	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{-\sqrt{3}}{2}$	$\frac{-\sqrt{3}}{3}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2
180º	π	0	–1	0	UNDEFINED	–1	UNDEFINED
210º	$\frac{7}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	–2
225º	$\frac{5}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240º	$\frac{4}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	–2	$\frac{-2\sqrt{3}}{3}$
270º	$\frac{3}{2}$	–1	0	UNDEFINED	0	UNDEFINED	–1
300º	$\frac{5}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\sqrt{3}$	2	$-\frac{2\sqrt{3}}{3}$
315º	$\frac{7}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	–1	–1	$\sqrt{2}$	$-\sqrt{2}$
330º	$\frac{11}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	–2
360º	2π	0	1	0	UNDEFINED	1	UNDEFINED

10 FIND THE VALUE OF EACH OF THE FOLLOWING:

A SIN (-120^0) **B** COS 600^0 **C** TAN (-300^0)

D CSC 990^0 **E** SEC 450^0 **F** COT (-420^0).

11 EVALUATE THE SIX TRIGONOMETRIC FUNCTIONS OF , IF IS IN STANDARD POSITION AND ITS TERMINAL SIDE CONTAINS THE GIVEN POINT P (x, y):

A P (5, 12) **B** P (– 7, 24) **C** P (5, – 6) **D** P (– 8, – 17)

E P (15, 8) **F** P (1, – 8) **G** P (– 3, –4) **H** P (0, 1)

12 LET BE AN ANGLE IN STANDARD POSITION. IDENTIFY THE QUADRANT IN WHICH LIES GIVEN THE FOLLOWING CONDITIONS:

A IF SIN < 0 AND COS < 0 **B** IF SIN > 0 AND TAN > 0

C IF SIN > 0 AND SEC < 0 **D** IF SEC > 0 AND COT < 0

E IF COS < 0 AND COT > 0 **F** IF SEC < 0 AND CSC > 0.

13 FIND THE ACUTE ANGLE , IF:

A $\text{SIN } 60^0 = \dfrac{1}{\text{CSC}}$ **B** SIN = COS **C** $\text{SIN } 70^0 = \text{COS}$

D $1 = \dfrac{\text{SIN}}{\text{COS } 80^0}$ **E** $\dfrac{\text{SIN}}{\text{CO}} = \text{COT } 35^0$ **F** $\dfrac{\text{SIN } 70^0}{\text{COS } 70^0} = \dfrac{\text{COS}}{\text{SIN}}$

14 IF IS OBTUSE AND $\text{COS} = \dfrac{-4}{5}$, FIND:

A SIN **B** TAN **C** CSC **D** COT.

15 IF $-90^0 <$ < 0 AND $\text{TAN} = -\dfrac{2}{3}$, FIND COS .

16 IN PROBLEMS A TO D BELOW, $\triangle ABC$ IS A RIGHT ANGLE TRIANGLE WITH $m(\angle C) = 90°$. LET a, b, c BE ITS SIDES WITH c THE HYPOTENUSE, a THE SIDE OPPOSITE ANGLE A AND b THE SIDE OPPOSITE ANGLE B. USING THE INFORMATION BELOW, FIND THE MISSING ELEME NTS OF EACH RIGHT TRIANGLE, ROUNDING ANSWERS CORRECT TO THE NEAREST WHOLE N

A $m(\angle B) = 60°$ AND $a = 18$ UNITS. **B** $m(\angle A) = 45°$ AND $c = 16$ UNITS.

C $m(\angle A) = 22°$ AND $b = 10$ UNITS. **D** $m(\angle B) = 52°$ AND $c = 47$ UNITS.

17 **A** FIND THE HEIGHT OF A TREE, IF THE ANGLE OF ELEVATION OF ITS TOP CHANGES FROM $25°$ TO $50°$ AS THE OBSERVER ADVANCES 15 METRES TOWARDS ITS BASE.

B THE ANGLE OF DEPRESSION OF THE TOP AND THE FOOT OF A FLAGPOLE AS SEEN FROM THE TOP OF A BUILDING 145 METRES AWAY ARE $25°$ AND $34°$ RESPECTIVELY. FIND THE HEIGHTS OF THE POLE AND THE BUILDING.

C TO THE NEAREST DEGREE, FIND THE ANGLE OF ELEVATION OF THE SUN WHEN A 9 METRE VERTICAL FLAGPOLE CASTS A SHADOW 3 METRES LONG.